New Approaches in Applied Statistics
Anuška Ferligoj and Andrej Mrvar (Editors)
Metodološki zvezki, 16, Ljubljana: FDV, 2000

Fuzzy Partitions

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Abstract

This paper gives an overview of several approaches to the definition of the notion of a fuzzy partition. Special attention is given to the fuzzy k-partitions which are results of the well known fuzzy k-means algorithm. Some methods for quantitative evaluation, comparison and combination of fuzzy k-partitions are suggested.

1 Introduction

Classification is a key issue in reasoning, learning, and decision making and is one of the most important of human activities. In the real world, classes are often inexact and categories vague. Researchers, especially in social and other human sciences, often have to deal with information expressed in linguistic terms. Using natural language, one cannot partition objects into nicely bounded classes. For example, the very natural concept of *good student* can hardly be used as a criterion for partitioning a group of students into two distinct subgroups of *good* and *not good* students.

A natural concept, e.g., the concept of a *good student*, can be thought of as associated with a collection of objects. Belongingness of an object in a collection is a matter of degree. Mathematical description of *belongingness*, or membership of objects in a collection, was introduced by Lotfi Zadeh (1965) in the theory of fuzzy sets. A fuzzy set A in a given set X is characterized by a membership function, which associates with each element of X a real number in the interval $[0, 1]$, with values of this function representing *grade of membership* of the corresponding element in A. Because full membership and full nonmembership in the fuzzy set can still be indicated by the values of 1 and 0, respectively, we can consider the concept of

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a classical (hard, crisp, nonfuzzy) set to be a restricted case of the more general concept of fuzzy set.

A fuzzy partition is a special collection of fuzzy sets. There is an abundant amount of literature dealing with the theory of fuzzy sets. Recent up-to-date monographs include Kruse et al. (1994), Klir and Yuan (1995), Grabisch et al. (1995), Hung and Walker (1997) among others. Studies concerned with the mathematical properties of fuzzy partitions are still rare.

First of all we will present several approaches to the definition of the notion of a fuzzy partition (Ruspini, 1969; Butnariu, 1983; Dubois and Prade, 1990; Hung and Walker, 1997). In practice, a fuzzy partition is a result of a search for structure in a data set. Techniques of search (pattern recognition) are applicable to medical records, psychological profiles, demographic features, aerial photos, etc. The result is a partition of objects into more or less homogeneous subgroups on the basis of an often subjectively chosen measure of similarity. Both the diversity of pattern recognition techniques and the number of different scientific disciplines in which these have been developed are striking. An excellent review of probabilistic, fuzzy and neural models for pattern recognition is in Bezdek (1993). The first fuzzy clustering algorithm was developed by Ruspini (1969). Dunn (1973) proposed a fuzzy k–means algorithm, which was generalized by Bezdek (1981). This algorithm results in the fuzzy k–partition of a data set. The number of fuzzy classification techniques has been growing, but the fuzzy k–means algorithm is still undoubtedly the most popular.

In Section 3 of this paper we focus on comparison of fuzzy k-partitions. Fuzzy partitions can be compared with respect to an appropriate quantitative characterization or can be ordered (partially ordered) according to a meaningful relation of complete or partial order. We propose a partial ordering of fuzzy k-partitions based on their sharpness (fuzziness) and a partial ordering of fuzzy k-partitions based on their m-importance. In general, the m-importance of a fuzzy partition U of a set X is determined by the values and the locations of the maximal coefficients of membership of object $x_j \in X$, $j \in \{1, \ldots, n\} = N_m$ in fuzzy clusters $u_i \in U$, for $i \in \{1, ..., k\} = N_k$. We also present several quantitative characterizations of fuzzy k-partitions and we discuss how they can be used in order to measure the amount of sharpness (fuzziness) or the amount of m-importance of a fuzzy partition.

In the fourth Section of this paper we consider a finite collection $\mathcal{S}$ of fuzzy k-partitions of the same set of objects (obtained, e.g., by applying different classification procedures to the same data, or because of different opinions of experts classifying objects). Then we will show how to find a lower and an upper bound of $\mathcal{S}$ with respect to the relation of m-importance. We will define α-equivalence of fuzzy k-partitions and we will present a method which allows one to find a lower and an upper bound of a collection of α-equivalent fuzzy k-partitions with respect to their sharpness.

2 The notion of a fuzzy partition

Let X be a nonempty set. A fuzzy set A in X is a function

$$\mu_A : X \to [0,1]. \tag{2.1}$$

The family of fuzzy set on X will be denoted by $\mathcal{F}(X)$. When we want to exhibit an element $x \in X$ that typically belongs to a fuzzy set A, we may demand its membership value to be greater than some threshold $\alpha \in (0,1]$. For each $\alpha \in (0,1]$ the α-level set A (or α-cut of A) is defined as the crisp set A_α with characteristic function

$$\mu_{A_\alpha}(x) = 1 \qquad \text{if } \mu_A(x) \geq \alpha, \tag{2.2}$$

$$\mu_{A_\alpha}(x) = 0 \qquad \text{otherwise.} \tag{2.3}$$

Then the membership function of a fuzzy set A can be expressed in terms of characteristic functions of its α-cuts according to the formula

$$\mu_A(x) = \sup_{\alpha \in (0,1]} \min\{\alpha, \mu_{A_\alpha}(x)\}. \tag{2.4}$$

The operations on $\mathcal{F}(X)$ are defined by the triangular norms, t-norms and t-conorms (Schweitzer and Sklar, 1960). The following t-norms are frequently used as fuzzy intersections:

- $T_0(x,y) = \min(x,y)$,
- $T_1(x,y) = x.y$,
- $T_\infty(x,y) = \max(x+y-1, 0)$,
- $T_s(x,y) = \log_s(1 + \frac{(s^x-1)(s^y-1)}{s-1})$, $s > 0, s \neq 1$.

The following t-conorms are frequently used as fuzzy unions:

- $S_0(x,y) = \max(x,y)$,
- $S_1(x,y) = x + y - x.y$,
- $S_\infty(x,y) = \min(x+y, 1)$,

- $S_s(x,y) = 1 - \log_s(1 + \frac{(s^{1-x}-1)(s^{1-y}-1)}{s-1}), s > 0, s \neq 1.$

Several definitions have been proposed for the concept of a fuzzy partition. According to Ruspini (1969) a k-tuple $(A_1, \ldots, A_k)$ of fuzzy sets from $\mathcal{F}(X)$ such that $\emptyset \neq A_i \neq X$ for all $i \in N_k$ and $\sum_{i=1}^{k} \mu_{A_i}(x) = 1$ for all $x \in X$ is called a fuzzy partition of X.

The following alternative definition of a fuzzy partition of a fuzzy set has been introduced by Butnariu (1983):

Let C be a fuzzy set on X. The family $A_1, \ldots, A_k$ of fuzzy sets from $\mathcal{F}(X)$ is a finite fuzzy partition of C if the next conditions are satisfied:

$$(\cup_{i=1}^{j}) \cap A_{j+1} = \emptyset, j \in N_{k-1}, \tag{2.5}$$

$$\cup_{i=1}^{n} A_j = C, \tag{2.6}$$

where $(A_i \cap A_j)(x) = \max(A_i(x) + A_j(x) - 1, 0)$ for all $x \in X$,
and $(A_i \cup A_j)(x) = \min(A_i(x) + A_j(x), 1)$ for all $x \in X$.

The equivalence of Ruspini's and Butnariu's definition is stated by the following theorem (Butnariu, 1983):

Theorem 1
The family $A_1, \ldots, A_k$ of fuzzy sets is a finite partition of a set C, i.e. it satisfies (5) and (6), if and only if $\sum_{i=1}^{k} \mu_{A_i}(x) = C(x)$ for all $x \in X$.

Dumitrescu (1992) showed that operations $\cup$ and $\cap$ given by S_∞ and T_∞ respectively are the only t-conorm and t-norm for which the Theorem 1 holds.

Some authors, e.g., Hung and Walker (1997), require that fuzzy sets $A_1, \ldots A_k$ in Ruspini's definition of a fuzzy partition must be normal fuzzy sets, i.e. for each A_i there is $x \in X:\ A_i(x) = 1$.

There has been also an attempt to study fuzzy partitions in metric spaces, conducted by Bouchon and Cohen (1986).

Dubois and Prade (1990) introduced a weak fuzzy partition as follows:
A k-tuple $(A_1, \ldots, A_k)$ of fuzzy sets from $\mathcal{F}(X)$ forms a weak partition of X if

$$\text{for all } i: \inf_x \max_i \{\mu_{A_i}(x)\} > 0, \tag{2.7}$$

while

$$\text{for all } i, j, i \neq j: \sup_x \min\{\mu_{A_i}(x), \mu_{A_j}(x)\} < 1. \tag{2.8}$$

In applications the bound 1 in (2.8) is usually replaced by 0.5 (Kruse et al., 1994).

Bezdek (1981) used Ruspini's definition of a fuzzy partition and devoted many of his papers to methods of fuzzy clustering and fuzzy pattern recognition where the results are *fuzzy k-partitions*. He defined partition spaces on a finite set X as follows:

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a given set of objects. Fix the integer k, $2 \leq k < n$ and denote by V_{kn} the usual vector space of real $k \times n$ matrices. Then: Fuzzy k-partition space associated with X:

$$P_{fk} = \{U \in V_{kn}; u_{ij} \in [0,1]; \sum_i u_{ij} = 1 \text{ for all } j; \sum_j u_{ij} > 0 \text{ for all } i\}. \qquad (2.9)$$

Here u_{ij} is the grade of membership of object $x_j \in X$ in fuzzy cluster u_i. Hard k-partition space associated with X:

$$P_k = \{U \in V_{kn}; u_{ij} \in \{0,1\}; \sum_i u_{ij} = 1 \text{ for all } j; \sum_j u_{ij} > 0 \text{ for all } i\}. \qquad (2.10)$$

If we relax the condition $\sum_j u_{ij} > 0$ we will get degenerate partitions. Degenerate fuzzy k-partition space associated with X:

$$P_{fko} = \{U \in V_{kn}; u_{ij} \in [0,1]; \sum_i u_{ij} = 1 \text{ for all } j\}. \qquad (2.11)$$

Degenerate hard k-partition space associated with X:

$$P_{ko} = \{U \in V_{kn}; u_{ij} \in \{0,1\}; \sum_i u_{ij} = 1 \text{ for all } j\}. \qquad (2.12)$$

It is obvious that $P_k \subset P_{fk} \subset P_{fko}$ and $P_k \subset P_{ko} \subset P_{fko}$.

Klir and Yuan (1995) call a fuzzy k-partition defined by Bezdek a fuzzy pseudopartition of X. They use the term fuzzy partition only for the result of a clustering method based on a fuzzy equivalence relation.

A fuzzy binary relation $\boldsymbol{E}$ in $X \times X$ is a fuzzy set defined on $X \times X$. $\boldsymbol{E}$ is reflexive if and only if for all $x \in X : \mu_E(x,x) = 1$. $\boldsymbol{E}$ is symmetric if and only if for all $x \in X$: $\mu_E(x,y) = \mu_E(y,x)$. $\boldsymbol{E}$ is transitive if $\mu_E(x,z) \geq \max_{y \in Y} \min\{\mu_E(x,y), \mu_E(y,z)\}$ is satisfied for each pair $(x,z) \in X \times X$. A fuzzy binary relation that is reflexive, symmetric and transitive is known as fuzzy equivalence relation or similarity relation. According to Klir and Yuan (1995) a fuzzy partition is a family of crisp partitions induced by the α-cuts of a fuzzy equivalence relation.

Example 1

Let $X = \{x_1, x_2, x_3, x_4\}$. The following partitions U and V are partitions of X from P_{f3o}:

$$U = \begin{pmatrix} 0.8 & 0.5 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.1 & 0.4 \\ 0.1 & 0.0 & 0.3 & 0.4 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1.0 & 0.6 & 0.7 & 0.1 \\ 0.0 & 0.3 & 0.0 & 0.5 \\ 0.0 & 0.1 & 0.3 & 0.4 \end{pmatrix}.$$

Example 2

Let $X = \{x_1, x_2, x_3, x_4\}$. Let the similarity among objects from X be given by the fuzzy equivalence relation $\boldsymbol{E}$:

$$E = \begin{pmatrix} 1.0 & 0.5 & 0.7 & 0.5 \\ 0.5 & 1.0 & 0.5 & 0.9 \\ 0.7 & 0.5 & 1.0 & 0.5 \\ 0.5 & 0.9 & 0.5 & 1.0 \end{pmatrix}.$$

Relation $\boldsymbol{E}$ induces four crisp partitions of its α-cuts.
$P_1: \ \alpha \in (0.0, 0.5]: \ \{\{x_1, x_2, x_3, x_4\}\}$,
$P_2: \ \alpha \in (0.5, 0.7]: \ \{\{x_2, x_4\}, \{x_1, x_3\}\}$,
$P_3: \ \alpha \in (0.7, 0.9]: \ \{\{x_2, x_4\}, \{x_1\}, \{x_3\}\}$,
$P_4: \ \alpha \in (0.9, 1.0]: \ \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}$.
The above sequence of partitions forms a crisp hierarchical partition of X.

In the next two Sections we will focus on characterization, comparison and combination of fuzzy k-partitions from P_{fko}.

3 Comparison of fuzzy partitions

Let $U, V \in P_{fko}$. We say that $U = V$ if and only if $u_{ij} = v_{ij}$ for all i, j. We want to introduce a relation of partial order on P_{fko} which allows one to order some pairs of $U, V \in P_{fko}$. We will use a modification of relation sharpness originally introduced by De Luca and Termini (1972) for fuzzy sets.

Definition 1

Let $U, V \in P_{fko}$ and $\alpha \in [0, 1]$. We say that U is α-sharper than V denoted by $U \prec_\alpha V$ if and only if

$$u_{ij} \leq v_{ij} \text{ for } v_{ij} < \alpha, \quad (3.1)$$

$$u_{ij} \geq v_{ij} \text{ for } v_{ij} \geq \alpha. \quad (3.2)$$

Relation $\prec_\alpha$ satisfies the following properties:

1. P_{fko} is partially ordered by $\prec_\alpha$.

2. The set $MIN_\alpha = \{U \in P_{fko} : u_{ij} > \alpha \text{ or } u_{ij} = 0 \text{ for all } i, j\}$ is the set of minimal elements of P_{fko} with respect to $\prec_\alpha$.
   It is obvious that when α approaches 1, then MIN_α approaches P_{ko}.

Relation α-sharpness captures the intuitive idea that the closer the coefficients u_{ij} are to 1 or 0, the sharper (less fuzzy, less diverse, less uncertain) is the partition U.

Example 3
Let $X = \{x_1, x_2, x_3, x_4\}$. Let $U, V \in P_{f30}$ be given by matrices

$$U = \begin{pmatrix} 0.6 & 0.0 & 0.4 & 0.1 \\ 0.2 & 1.0 & 0.5 & 0.2 \\ 0.2 & 0.0 & 0.1 & 0.7 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0.5 & 0.0 & 0.4 & 0.2 \\ 0.3 & 0.8 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Then $U \prec_\alpha V$ for α=0.4.

Suppose that classes $C_1, \ldots, C_k$ are arranged according to their logical importance from the most important to the least important (e.g., arrangement of k medical diagnoses from the most severe (C_1) to the least severe (C_k)). Suppose that we have two fuzzy partitions of the same set of objects (e.g., the same set of patients) into classes $C_1, \ldots, C_k$. We will consider the fuzzy partition U more important than the fuzzy partition V if for each object (patient) x_j either 1. or 2. holds:

1. the maximum of the set $\{u_{ij}, v_{ij}\}$ over classes C_i where $u_{ij} \neq v_{ij}$, is a coefficient from U,
2. $\max\{u_{ij}, v_{ij}\}$ over classes C_i where $u_{ij} \neq v_{ij}$ is the element $u_{rj} \in U$ which is equal to the element $v_{sj} \in V$, but u_{rj} is in the more important class ($r < s$).

We define the relation m-inclusion on $P_{fko} \times P_{fko}$ as follows:

Definition 2
Let $U, V \in P_{fko}$. For each $x_j \in X$, let $I_j = \{i : u_{ij} \neq v_{ij}\}, u_{rj} = \max_{i \in I_j}\{u_{ij}\}$ and $v_{sj} = \max_{i \in I_j}\{v_{ij}\}$. Then U is m-included in V, denoted by $U \sqsubset_m V$, if and only if for each $x_j \in X$ either $I_j = \emptyset$, or $u_{rj} > v_{sj}$; or $u_{rj} = v_{sj}$ and $r < s$.

Relation $\sqsubset_m$ satisfies the following properties:

1. P_{fko} is partially ordered by $\sqsubset_m$.
2. Poset $(P_{fko}, \sqsubset_m)$ forms a lattice.

Example 4
Let $U, V \in P_{f3o}$ be partitions of $X = \{x_1, x_2, x_3, x_4\}$ given by matrices

$$U = \begin{pmatrix} 0.0 & 0.6 & 0.1 & 0.2 \\ 1.0 & 0.3 & 0.5 & 0.0 \\ 0.0 & 0.1 & 0.4 & 0.8 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0.6 & 0.0 & 0.3 & 0.0 \\ 0.4 & 0.6 & 0.5 & 0.7 \\ 0.0 & 0.4 & 0.2 & 0.3 \end{pmatrix}.$$

Then $U \sqsubset_m V$.

Theorem 2
Let $U, V \in P_{fko}$ and $\alpha \in (0,1)$. If $U \prec_\alpha V$ then $U \sqsubset_m V$.

Fuzzy partitions can be compared with respect to some quantitative characterizations which are defined as real functions $g : P_{fko} \to [0, \infty)$. We will show that some of these characterizations can be used in order to measure the amount of α-sharpness or the amount of m-importance of a fuzzy partition.

Definition 3
Function $g : P_{fko} \to R_0^+$ is called a measure of α-sharpness if for $U, V \in P_{fko}$ such that $U \prec_\alpha V$, we have $g(U) \leq g(V)$.

If $\alpha = \frac{1}{k}$ then a measure of α-sharpness becomes a convenient way of measuring fuzziness of a fuzzy partition.

Example 5
Several quantitative characterizations of fuzzy partitions suggested in Bezdek (1981) can be used for constructing measures of α-sharpness. For example:

a) Degree of separation

$$Z(U) = 1 - \max_j \{\min_i u_{ij}\}. \tag{3.3}$$

Proof: If $U \prec_\alpha V$ then $\min_i u_{ij} \leq \min_i v_{ij}$, therefore $Z(U) \geq Z(V)$ and $g_1(U) = 1 - Z(U) \leq 1 - Z(V) = g_1(V)$, i.e. $g_1(U)$ is a measure of α-sharpness of U.

b) Partition coefficient

$$F(U) = \frac{1}{n} \sum_i \sum_j u_{ij}^2. \tag{3.4}$$

Proof: If $U \prec_\alpha V$ then $\sum_i \sum_j (u_{ij} - \alpha)^2 \geq \sum_i \sum_j (v_{ij} - \alpha)^2$, it means
$\sum_i \sum_j u_{ij}^2 - 2\sum_j \sum_i u_{ij}\alpha + nk\alpha^2 \geq \sum_i \sum_j v_{ij}^2 - 2\sum_j \sum_i v_{ij}\alpha + nk\alpha^2$, it means
$\sum_i \sum_j u_{ij}^2 - 2n\alpha + nk\alpha^2 \geq \sum_i \sum_j v_{ij}^2 - 2n\alpha + nk\alpha^2$, therefore $\sum_i \sum_j u_{ij}^2 \geq \sum_i \sum_j v_{ij}^2$, and $g_2(U) = 1 - F(U)$ is a measure of α-sharpness of U.

c) Partition entropy

$$H(U) = -\frac{1}{n} \sum_i \sum_j u_{ij} \log_a u_{ij} \tag{3.5}$$

where $a \in (1, \infty)$ and $u_{ij} \log_a u_{ij} = 0$ for $u_{ij} = 0$.
Proof:
We will show that if $V \prec_\alpha U$ then $H(V) \leq H(U)$.
Let $x_j \in X$ and $I(u_j) = \{i : u_{ij} > 0\}$. Let us consider function $h : [0,1] \to R$ defined by $h(x) = -x. \log_a x$, for $x \in (0,1], a \in (1, \infty), h(0) = 0$.

Then $\frac{dh}{dx} = -\log_a x - \frac{1}{\log a}$ and $\frac{d^2h}{dx} \leq 0$. Hence $h(x+\delta) \leq h(x) + \delta\frac{dh}{dx}$ for all $\delta \in (0,1)$.

Let us consider $u(x_j) \in U \in P_{fko}$ such that $u_{ij} < \alpha$ for $i \leq r$ and $u_{ij} \geq \alpha$ for $i > r$.

Then $v_{ij} = u_{ij} + \delta_i$, where $\delta_i \leq 0$ for $i \leq r$ and $\delta_i \geq 0$ for $i > r$.

Obviously, $\sum_{i=1}^{r}(-\delta_i) = \sum_{i=r+1}^{k}\delta_i$, and $\log_a u_{ij} \leq \log_a \alpha$ for $i \leq r$ and $\log_a u_{ij} \geq \log_a \alpha$ for $i > r$.

Therefore

$$\begin{aligned} h(v(x_j)) &= \sum_{i}^{k} -v_{ij}.\log_a v_{ij} = -\sum_{i}^{k}(u_{ij}+\delta_i).\log_a(u_{ij}+\delta_i) \\ &\leq -\sum_{i}^{k} u_{ij}.\log_a u_{ij} + \sum_{i}^{k}\delta_i(-\log_a u_{ij} - \frac{1}{\log a}) \\ &\leq h(u(x_j)) + (\log_a \alpha + \frac{1}{\log a}).\sum_{i=1}^{r}(-\delta_i) \\ &\quad + \sum_{i=r+1}^{k}\delta_i(-\log_a u_{ij} - \frac{1}{\log a}) \\ &= h(u(x_j)) + \sum_{i=r+1}^{k}\delta_i(\log_a \alpha - \log_a u_{ij}) \leq h(u(x_j)). \end{aligned}$$

Hence $H(V) = \sum_j h(v_{ij}) \leq \sum_j h(u_{ij}) = H(U)$, Q.E.D.

Example 6

Consider fuzzy partitions U, V from Example 3. $U \prec_\alpha V$ for $\alpha = 0.4$. Let g_1, g_2 and H be measures of α-sharpness introduced in Example 5. Let $a = 10$ in definition of H. Then:

$g_1(U) = 0.2$, $g_2(U) = 0.4$, $H(U) = .293$, and $g_1(V) = 0.2$, $g_2(V) = 0.535$, $H(V) = .384$.

A simple way of constructing measures of α-sharpness of fuzzy partitions is based on α-combination of fuzzy clusters.

Definition 4

Let $u_r, u_s \in \mathcal{F}(X)$ and $\alpha \in (0,1)$. Minimal α-combination of u_r, u_s is the fuzzy set $u_p \in \mathcal{F}(X)$ defined by:
for all $x \in X$:

$$\begin{aligned} u_p(x) &= \min\{u_r(x), u_s(x)\} \textit{ if } \min\{u_r(x), u_s(x)\} < \alpha, \\ &= 0 \textit{ otherwise.} \end{aligned} \tag{3.6}$$

We denote $u_p = u_{r\underline{\alpha}}u_s$.

Definition 5
Let $u_r, u_s \in \mathcal{F}(X)$ and $\alpha \in (0,1)$. Maximal α-combination of u_r, u_s is the fuzzy set $u_q \in \mathcal{F}(X)$ defined by:
for all $x \in X$:

$$\begin{aligned} u_q(x) &= \max\{u_r(x), u_s(x)\} \text{ if } \max\{u_r(x), u_s(x)\} \geq \alpha, \\ &= 1 \text{ otherwise.} \end{aligned} \tag{3.7}$$

We denote $u_q = u_r^{\overline{\alpha}} u_s$.

It is easy to prove that for $U, V \in P_{fko}$ such that $U \prec_\alpha V$, for all $(r \times s) \in N_k \times N_k$:

$$0 \leq u_p(x) = (u_{r_{\underline{\alpha}}} u_s)(x) \leq (v_{r_{\underline{\alpha}}} v_s)(x) = v_p(x) < \alpha, \tag{3.8}$$

$$\alpha \leq v_q(x) = (v_r^{\overline{\alpha}} v_s)(x) \leq (u_r^{\overline{\alpha}} u_s)(x) = u_q(x) \leq 1. \tag{3.9}$$

Theorem 3
Let $f : [0, \alpha] \to R_0^+$ such that f is nondecreasing on $[0, \alpha]$. Then function $g : P_{fko} \to R_0^+$ defined by

$$g(U) = \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} \sum_{x} f((u_{r_{\underline{\alpha}}} u_s)(x)) \tag{3.10}$$

is a measure of α-sharpness of U.

Theorem 4
Let $f : [\alpha, 1] \to R_0^+$ such that f is nonincreasing on $[\alpha, 1]$. Then function $g : P_{fko} \to R_0^+$ defined by

$$g(U) = \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} \sum_{x} f((u_r^{\overline{\alpha}} u_s)(x)) \tag{3.11}$$

is a measure of α-sharpness of U.

Example 7
Let $U, V \in P_{f3o}$ be partitions of $X = \{x_1, x_2, x_3, x_4\}$ given by matrices

$$U = \begin{pmatrix} 0.8 & 0.4 & 1.0 & 0.3 \\ 0.1 & 0.2 & 0.0 & 0.2 \\ 0.1 & 0.4 & 0.0 & 0.5 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0.75 & 0.40 & 0.70 & 0.30 \\ 0.15 & 0.25 & 0.20 & 0.25 \\ 0.10 & 0.35 & 0.10 & 0.45 \end{pmatrix}.$$

$U \prec_\alpha V$ for $\alpha = 0.3$.
Let $f_1 : [0, 0.3] \to R_0^+$ is an identity function, i.e. $f_1(x) = x$ for all $x \in [0, 0.3]$. Then $g_1(U) = \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} \sum_x (u_{r_{\underline{0.3}}} u_s)(x)$ is a measure of α-sharpness of U for $\alpha = 0.3$.
It is clear that

$(u_{1_{\underline{0.3}}}u_2)(x_1) = 0.1, (u_{1_{\underline{0.3}}}u_2)(x_2) = 0.2, (u_{1_{\underline{0.3}}}u_2)(x_3) = 0.0, (u_{1_{\underline{0.3}}}u_2)(x_4) = 0.2,$
$(u_{1_{\underline{0.3}}}u_3)(x_1) = 0.1, (u_{1_{\underline{0.3}}}u_3)(x_2) = 0.0, (u_{1_{\underline{0.3}}}u_3)(x_3) = 0.0, (u_{1_{\underline{0.3}}}u_3)(x_4) = 0.0,$
$(u_{2_{\underline{0.3}}}u_3)(x_1) = 0.1, (u_{2_{\underline{0.3}}}u_3)(x_2) = 0.2, (u_{2_{\underline{0.3}}}u_3)(x_3) = 0.0, (u_{2_{\underline{0.3}}}u_3)(x_4) = 0.2.$
Therefore $g_1(U) = 1.1$.
Analogously we can find out that $g_1(V) = 1.75$.

Let $f_2 : [0.3, 1] \to R_0^+$ is defined by $f_1(x) = 1 - x$ for all $x \in [0.3, 1]$.
Then $g_2(U) = nk - \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} \sum_x (u_r^{\overline{0.3}} u_s)(x)$ is a measure of α-sharpness of U for $\alpha = 0.3$.
It is clear that
$(u_1^{\overline{0.3}}u_2)(x_1) = 0.8, (u_1^{\overline{0.3}}u_2)(x_2) = 0.4, (u_1^{\overline{0.3}}u_2)(x_3) = 1.0, (u_1^{\overline{0.3}}u_2)(x_4) = 0.3,$
$(u_1^{\overline{0.3}}u_3)(x_1) = 0.8, (u_1^{\overline{0.3}}u_3)(x_2) = 0.4, (u_1^{\overline{0.3}}u_3)(x_3) = 1.0, (u_1^{\overline{0.3}}u_3)(x_1) = 0.5,$
$(u_2^{\overline{0.3}}u_3)(x_1) = 1.0, (u_2^{\overline{0.3}}u_3)(x_2) = 0.4, (u_2^{\overline{0.3}}u_3)(x_3) = 1.0, (u_2^{\overline{0.3}}u_3)(x_4) = 0.5.$
Therefore $g_2(U) = 12 - 8.1 = 3.9$.
Analogously we can find out that $g_2(V) = 4.75$.

Now we will provide some examples of measures of m-importance.

Example 8
Let $U \in P_{fko}$. Then

$$\psi_1(U) = n - \sum_j \max_i u_{ij} \tag{3.12}$$

is a measure of m-importance.
Proof: If $U \sqsubset_m V$ then $\max_i u_{ij} \geq \max_i v_{ij}$ for all j, therefore $\psi_1(U) \leq \psi_1(V)$, and ψ_1 is a measure of m-importance.

Also

$$\psi_2(U) = 1 - \Pi_j \max_i u_{ij} \tag{3.13}$$

is a measure of m-importance.
The proof is obvious.

Example 9
Consider fuzzy partitions U, V from Example 4. Let ψ_1, ψ_2 be measures of m-importance introduced in Example 8. Then
$\psi_1(U) = 4 - (1 + 0.6 + 0.5 + 0.8) = 1.1$ and $\psi_1(V) = 4 - (0.6 + 0.6 + 0.5 + 0.7) = 1.6$, and $\psi_2(U) = 1 - 0.24 = .76$, $\psi_2(V) = 1 - 0.126 = 0.874$.

4 Combination of fuzzy partitions

Let $\mathcal{S} = \{U_1, \ldots, U_m\}$ be a collection of fuzzy k-partitions and let $\boldsymbol{R}$ be a relation of partial order on P_{fko}. We want to find a fuzzy partition $W \in P_{fko}$ such that

W $\boldsymbol{R}$ U_t for all $t \in N_m$. W will be called a lower bound of $\mathcal{S}$ with respect to $\boldsymbol{R}$. Analogously, we want to find a fuzzy partition $Z \in P_{fko}$ such that U_t $\boldsymbol{R}$ Z for all $t \in N_m$. Z will be called an upper bound of $\mathcal{S}$ with respect to $\boldsymbol{R}$.

Let $\boldsymbol{R}$ be the relation of m-importance introduced in Section 3. Since $(P_{fko}, \sqsubset_m)$ forms a lattice, a lower and an upper bound exist for any collection $\{U_1, \ldots, U_m\}$ of fuzzy partitions from P_{fko}. We will describe an algorithm for construction of these bounds.

Let $U(x_j)$ denote the column of matrix U which includes coefficients of membership of object $x_j \in X$ in fuzzy clusters $u_i, i \in N_k$. Let $V(x_j)$ denote the column of matrix V which includes coefficients of membership of object $x_j \in X$ in fuzzy clusters $v_i, i \in N_k$. Let $I_j = \{i \in N_k : u_i(x_j) \neq v_i(x_j)\}$, $u_{rj} = \max_{i \in I_j}\{u_{ij}\}$ and $v_{sj} = \max_{i \in I_j}\{v_{ij}\}$. Then $U(x_j) \sqsubset V(x_j)$ if and only if either $I_j = \emptyset$, or $u_{rj} > v_{sj}$, or $u_{rj} = v_{sj}$ and $r < s$. It is obvious that $U \sqsubset_m V$ if and only if $U(x_j) \sqsubset V(x_j)$ for all $x_j \in X$.

The following algorithm leads to a lower bound W and an upper bound Z for a collection of fuzzy k-partitions $\{U_1, \ldots, U_m\}$ with respect to their m-importance.

Algorithm 1
Input: $U, V \in P_{fko}$.
Step 1. Put $t = 1$. For all $j \in N_n :$ $W(x_j) := U_t(x_j)$ and $Z(x_j) := U_t(x_j)$.
Step 2. Put $t := t + 1$. If $t = m$ then stop. Else go to Step 3.
Step 3. If $U_t(x_j) \sqsubset W(x_j)$ then $W(x_j) := U_t(x_j)$.
If $Z(x_j) \sqsubset U_t(x_j)$ then $Z(x_j) := U_t(x_j)$.
Go to Step 2.

Example 10
Let $U_1, U_2, U_3 \in P_{f3o}$ be fuzzy partitions of $X = \{x_1, x_2, x_3, x_4\}$ given by matrices

$$U_1 = \begin{pmatrix} 0.4 & 0.0 & 0.3 & 0.0 \\ 0.6 & 0.7 & 0.3 & 0.0 \\ 0.0 & 0.3 & 0.4 & 1.0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0.8 & 0.2 & 0.3 & 0.9 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.6 & 0.0 \end{pmatrix}$$

and

$$U_3 = \begin{pmatrix} 0.5 & 0.1 & 0.2 & 0.0 \\ 0.3 & 0.7 & 0.2 & 1.0 \\ 0.2 & 0.2 & 0.6 & 0.0 \end{pmatrix}.$$

Then, applying Algorithm 1, we will get the lower bound W and the upper bound Z of collection $\{U_1, U_2, U_3\}$ with respect to the relation m-sharpness as follows:

$$W = \begin{pmatrix} 0.8 & 0.0 & 0.3 & 0.0 \\ 0.1 & 0.7 & 0.1 & 1.0 \\ 0.1 & 0.3 & 0.6 & 0.0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0.5 & 0.1 & 0.3 & 0.9 \\ 0.3 & 0.7 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.4 & 0.0 \end{pmatrix}$$

It is obvious that $W \sqsubset_m U_t$ and $U_t \sqsubset_m Z$ for all $t \in N_3$.

In general, the lower and the upper bound for a collection of fuzzy k-partitions with respect to relation α-sharpness do not exist. However, we might be able to find these bounds for a collection of α-equivalent fuzzy partitions.

Definition 6

Let $\alpha \in [0,1]$ and let $U \in P_{fko}$. The α-cut of fuzzy partition U is characterized by the $k \times n$ matrix U^α defined as follows:

$$u_{ij}^\alpha = 1 \qquad \text{if } u_{ij} \geq \alpha, \tag{4.1}$$

$$u_{ij}^\alpha = 0 \qquad \text{otherwise.} \tag{4.2}$$

Definition 7

Let $\alpha \in [0,1]$. We will say that fuzzy partitions $U, V \in P_{fko}$ are α-equivalent if $U^\alpha = V^\alpha$.

The following hold:

1. All fuzzy partitions are α-equivalent at $\alpha = 0$.
2. $U^\alpha = V^\alpha$ for all $\alpha \in [0,1]$ if and only if $U = V$.

Theorem 5

Let $U, V \in P_{fko}$ be such that $U \prec_\alpha V$. Then $U^\alpha = V^\alpha$.

The proof is obvious.

It is easy to check that there are two trivial cases when $U_1^\alpha = U_2^\alpha = \ldots = U_m^\alpha$. Let $\delta_1 = \min_{tij}\{u_{(t)ij}\}$, and $\delta_2 = \max_{tij}\{u_{(t)ij}\}$, $t \in N_m$. Then for $\alpha \in [0, \delta_1]$ we get $U_t^\alpha = [1]$, and for $\alpha \in (\delta_2, 1]$ we get $U_t^\alpha = [0]$ for all $t \in N_m$. We want to find a nontrivial α, i.e. $\alpha \in (\delta_1, \delta_2]$ such that $U_1^\alpha = U_2^\alpha = \ldots = U_m^\alpha$. We propose the following Algorithm:

Algorithm 2

Step 1. Put $\alpha_0 = \min_{jt}\{\max_i u_{(t)ij}\}$ and $\delta_1 = \min_{tij}\{u_{(t)ij}\}$.

Step 2. Put $\alpha = \alpha_0$ and construct U_t^α for all $t \in N_m$.

Step 3. Find the set S of those elements $u_{(r)ij}$ from matrices $U_1, \ldots, U_m$ such that $u_{(r)ij}^\alpha = 0$ and there exists $s \in N_m$ such that $u_{(s)ij}^\alpha = 1$. If $S = \emptyset$ then stop.
Else go to Step 4.

Step 4. Put α_0=minimal element from S.

If $\alpha_0 \leq \delta_1$ then stop.
Else go to Step 2.

Example 11
Let $U_1, U_2, U_3 \in P_{f4o}$ be fuzzy partitions of $X = \{x_1, x_2, x_3, x_4, x_5\}$ given by matrices

$$U_1 = \begin{pmatrix} 0.50 & 0.10 & 0.40 & 0.15 & .40 \\ 0.40 & 0.05 & 0.08 & 0.42 & .50 \\ 0.00 & 0.45 & 0.12 & 0.43 & .05 \\ 0.10 & 0.40 & 0.40 & 0.00 & .05 \end{pmatrix}, \ U_2 = \begin{pmatrix} 0.50 & 0.10 & 0.45 & 0.15 & 0.50 \\ 0.45 & 0.00 & 0.08 & 0.45 & 0.40 \\ 0.00 & 0.40 & 0.07 & 0.40 & 0.05 \\ 0.05 & 0.50 & 0.40 & 0.00 & 0.05 \end{pmatrix}$$

and

$$U_3 = \begin{pmatrix} 0.48 & 0.10 & 0.54 & 0.15 & 0.50 \\ 0.42 & 0.10 & 0.01 & 0.40 & 0.50 \\ 0.05 & 0.40 & 0.05 & 0.40 & 0.00 \\ 0.05 & 0.40 & 0.40 & 0.05 & 0.00 \end{pmatrix}.$$

$\delta_1 = \min_{tij}\{u_{(t)ij}\} = 0$.
We want to find $\alpha > 0$ such that $U_1^\alpha = U_2^\alpha = U_3^\alpha$. We will use Algorithm 2 as follows:

Step 1. $\alpha_0 = \min\{0.50, 0.54, 0.45\} = 0.45$.
Step 2. $\alpha = 0.45$. Then

$$U_1^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ U_2^\alpha = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \ U_3^\alpha = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 3. $S = \{0.42, 0.40\}$.
Step 4. $\alpha_0 = 0.4$.
Step 2.

$$U_1^\alpha = U_2^\alpha = U_3^\alpha = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Step 3. $S = \emptyset$.
Therefore, we have that $U_1^\alpha = U_2^\alpha = U_3^\alpha$ for $\alpha = 0.4$.

Let $\mathcal{S} = \{U_1, \ldots, U_m\}$ be a collection of α-equivalent fuzzy k-partitions. For $x_j \in X$ let us denote $I_{0j} = \{i \in N_k : u^\alpha_{(t)ij} = 0\}, I_{1j} = \{i \in N_k : u^\alpha_{(t)ij} = 1\}$. Let us create a matrix $\underline{S} \in V_{kn}$ as follows:

$$\underline{s}_{ij} = \max_t\{u_{(t)ij}\} \text{ for all } i \in I_{1j}, \tag{4.3}$$
$$\underline{s}_{ij} = \min_t\{u_{(t)ij}\} \text{ for all } i \in I_{0j}. \tag{4.4}$$

It is obvious that for all $t \in N_m$ we get: $\underline{s}_{ij} \leq u_{(t)ij}$ for all $u_{(t)ij} < \alpha$ and $\underline{s}_{ij} \geq u_{(t)ij}$ for all $u_{(t)ij} \geq \alpha$. However, matrix S is not a matrix representing a fuzzy k-partition (in general, $\sum_i \underline{s}_{ij} \neq 1$.) We want to find a fuzzy partition $W \in P_{fko}$ such that $W \prec_\alpha U_t$ for all $t \in N_m$ and $\sum_i \sum_j (w_{ij} - \underline{s}_{ij})^2$ is minimal. The solution is described in the following method.

Method 1
Let $U_1, U_2, \ldots, U_m$ be partitions from P_{fko} which are α-equivalent at a nontrivial level of α. For $j \in N_m$ let us denote:
$I_{0j} = \{i \in N_k : u^\alpha_{(t)ij}) = 0\}, I_{1j} = \{i \in N_k : u^\alpha_{(t)ij} = 1\}, c_{0j} =$ card $I_{0j}, c_{1j} =$ card I_{1j},
$S_{0j} = \sum_{i \in I_{0j}} \min_t\{u_{(t)ij}\}$ and $S_{1j} = \sum_{i \in I_{1j}} \max_t\{u_{(t)ij}\}$.
Requirement: Let $S_{1j} \leq 1$ for all $j \in N_n$.
Then we can find $W \in P_{fko}$ such that $W \prec_\alpha U_t$ for $t \in N_m$ as follows: a) If $S_{0j} + S_{1j} \leq 1$ then for $i \in I_{0j}$:

$$w_{ij} = \min_t\{u_{(t)ij}\}, \tag{4.5}$$

for $i \in I_{1j}$:

$$w_{ij} = \max_t\{u_{(t)ij}\} + \frac{1 - (S_{0j} + S_{1j})}{c_{1j}}. \tag{4.6}$$

b) If $S_{0j} + S_{1j} > 1$ then for $i \in I_{1j}$:

$$w_{ij} = \max_t\{u_{(t)ij}\}. \tag{4.7}$$

For $i \in I_{0j}$ the values of w_{ij} are calculated as follows:

Algorithm 3

Step 1.
$I^0 := I_{0j};\ S := S_{0j} + S_{1j} - 1.$
Step 2.
$D = \frac{S}{\text{card } I^0}$, and $I^1 = \{i \in I^0 : \min_t\{u_{(t)ij}\} < D\}$.
Step 3.
If card $I^1 = 0$ then for all $i \in I^0 : w_{ij} = \min_t\{u_{(t)ij}\} - D$ and stop.
Else for all $i \in I^1 : w_{ij} = 0,\ S := S - \sum_{i \in I^1} \min_t\{u_{(t)ij}\}$,
$I^0 := I^0 - I^1$ and go to Step 2.

The proof that Method 1 leads to a fuzzy partitions W such that $W \prec_\alpha U_t$ for all $t \in N_m$ and $\sum_i \sum_j (w_{ij} - \underline{s}_{ij})^2$ is minimal is given in Appendix 1.

Example 12
Consider the fuzzy partitions from Example 11. We have shown that $U_1^\alpha = U_2^\alpha = U_3^\alpha$ for $\alpha = 0.4$. We will use Method 1 in order to find fuzzy partition $W \in P_{f4o}$ such that $U \prec_\alpha U_t$ for all $t \in N_3$ and $\alpha = 0.4$.

For $j = 1 : I_{0j} = \{3, 4\}, I_{1j} = \{1, 2\}, S_{0j} = 0.00+0.05 = 0.05, S_{1j} = 0.50+0.45 = 0.95$.
For $j = 2 :\ I_{0j} = \{1, 2\},\ I_{1j} = \{3, 4\},\ S_{0j} = 0.10 + 0.00 = 0.10, S_{1j} = 0.45 + 0.50 = 0.95$.
For $j = 3 :\ I_{0j} = \{2, 3\},\ I_{1j} = \{1, 4\},\ S_{0j} = 0.01 + 0.05 = 0.06, S_{1j} = 0.54 + 0.40 = 0.94$.
For $j = 4 :\ I_{0j} = \{1, 4\},\ I_{1j} = \{2, 3\},\ S_{0j} = 0.15 + 0.00 = 0.15, S_{1j} = 0.45 + 0.43 = 0.88$.
For $j = 5 :\ I_{0j} = \{3, 4\},\ I_{1j} = \{1, 2\},\ S_{0j} = 0.00 + 0.00 = 0.00, S_{1j} = 0.50 + 0.50 = 1.00$.
Therefore the requirement of Method 1 is satisfied.
Now we will construct columns of matrix W.
For $j = 1 :\ S_{0j} + S_{1j} = 0.05 + 0.95 = 1 \leq 1$, therefore
$w_{11} = 0.5, w_{21} = 0.45, w_{31} = 0, w_{41} = 0.05$.
For $j = 2 :\ S_{0j} + S_{1j} = 0.10 + 0.95 = 1.05 > 1$, therefore $w_{32} = 0.45, w_{42} = 0.50$.
In order to obtain w_{12} and w_{22} we need to use Algorithm 3.
Step 1. $I^0 := I_{02} = \{1, 2\}$ and $S := S_{02} + S_{12} - 1 = 0.05$.
Step 2. $D = \frac{0.05}{2} = 0.025$ and $I^1 = \{2\}$.
Step 3. Because $I^1 = \{2\}$, we get $w_{22} = 0$.
$S := 0.05 - 0 = 0.05, I^0 := I^0 - I^1 = \{1\}$.
Step 2. $D = \frac{0.05}{1} = 0.05$ and $I^1 = \emptyset$.
Step 3. $w_{12} = 0.10 - 0.05 = 0.05$.

The remaining columns of W are calculated analogously. Then

$$W = \begin{pmatrix} 0.50 & 0.05 & 0.54 & 0.12 & .50 \\ 0.45 & 0.00 & 0.01 & 0.45 & .50 \\ 0.00 & 0.45 & 0.05 & 0.43 & .00 \\ 0.05 & 0.50 & 0.40 & 0.00 & .00 \end{pmatrix}.$$

Let $\mathcal{S} = \{U_1, \ldots, U_m\}$ be a collection of α-equivalent fuzzy k-partitions. For $x_j \in X$ let us denote $I_{0j} = \{i \in N_k : u^\alpha_{(t)ij} = 0\}, I_{1j} = \{i \in N_k : u^\alpha_{(t)ij} = 1\}$. Let us create a matrix $\overline{S} \in V_{kn}$ as follows:

$$\overline{s}_{ij} = \min_t\{u_{(t)ij}\} \text{ for all } i \in I_{1j} \tag{4.8}$$

$$\overline{s}_{ij} = \max_t\{u_{(t)ij}\} \text{ for all } i \in I_{0j}. \tag{4.9}$$

It is obvious that for all $t \in N_m$ we get: $u_{(t)ij} \leq \overline{s}_{ij}$ for all $u_{(t)ij} < \alpha$ and $u_{(t)ij} \geq \overline{s}_{ij}$ for all $u_{(t)ij} \geq \alpha$. However, matrix $\overline{S}$ is not a matrix representing a fuzzy k-partition (in general, $\sum_i \overline{s}_{ij} \neq 1$.) We want to find a fuzzy partition $Z \in P_{fko}$ such that $U_t \prec_\alpha Z$ for all $t \in N_m$ and $\sum_i \sum_j (z_{ij} - \overline{s}_{ij})^2$ is minimal. The solution is described in the following method.

Method 2

Let $U_1, \ldots, U_m$ be partitions from P_{fko} which are α-equivalent at a nontrivial value of α. For $j \in N_n$ let us denote:

$I_{0j} = \{i \in N_k : u^\alpha_{(t)ij} = 0\}, I_{1j} = \{i \in N_k : u^\alpha_{(t)ij} = 1\}, c_{0j} =$ card $I_{0j}, c_{1j} =$ card I_{1j},

$S_{0j} = \sum_{i \in I_{0j}} \max_t\{u_{(t)ij}\}$ and $S_{1j} = \sum_{i \in I_{1j}} \min_t\{u_{(t)ij}\}$.

Requirements: Let $S_{0j} + \alpha.c_{1j} \leq 1$ for all $j \in N_n$. Let for $S_{0j} + S_{1j} \leq 1$ we have that $\frac{1-(S_{0j}+S_{1j})}{c_{0j}} < \alpha - \max_{i \in I_{0j}}\{u_{(t)ij}\}$ for all $j \in N_n$. Then we can find $Z \in P_{fko}$ such that $U_t \prec_\alpha Z$ for $t \in N_m$ as follows: a) If $S_{0j} + S_{1j} \leq 1$ then for $i \in I_{1j}$:

$$z_{ij} = \min_t\{u_{(t)ij}\}, \tag{4.10}$$

for $i \in I_{0j}$:

$$z_{ij} = \max_t\{u_{(t)ij}\} + \frac{1 - (S_{0j} + S_{1j})}{c_{0j}}. \tag{4.11}$$

b) If $S_{0j} + S_{1j} > 1$ then for $i \in I_{0j}$:

$$z_{ij} = \max_t\{u_{(t)ij}\}. \tag{4.12}$$

For $i \in I_{1j}$ the values of z_{ij} are calculated as follows:

Algorithm 4

Step 1

$I^1 := I_{1j}$; $S := S_{0j} + S_{1j} - 1$.

Step 2

$D = \frac{S}{\text{card } I^1}$, and $I^2 = \{i \in I^1 : \min_t\{u_{(t)ij}\} - D < \alpha\}$.

Step 3

If card $I^2 = 0$ then for all $i \in I^1 : z_{ij} = \min_t\{u_{(t)ij}\} - D$ and stop.

Else for all $i \in I^2 : z_{ij} = \alpha$, $S := S - \sum_{i \in I^2}(\min_t\{u_{(t)ij}\} - \alpha)$,

$I^1 := I^1 - I^2$ and go to Step 2.

The proof that Method 2 leads to a fuzzy partition Z such that $U_t \prec_\alpha Z$ for all $t \in N_m$ and $\sum_i \sum_j (z_{ij} - \overline{s}_{ij})^2$ is minimal is given in Appendix 2.

Example 13

Consider the fuzzy partitions from Example 11. We have shown that $U_1^\alpha = U_2^\alpha = U_3^\alpha$ for $\alpha = 0.4$. We will use Method 2 in order to find fuzzy partition $Z \in P_{f4o}$ such that $U_t \prec_\alpha Z$ for all $t \in N_3$ and $\alpha = 0.4$.
For $j = 1: \ I_{0j} = \{3,4\}, I_{1,j} = \{1,2\}, c_{0j} = c_{1j} = 2.$
$S_{0j} = 0.05 + 0.10 = 0.15, S_{1j} = 0.48 + 0.40 = 0.88, S_{0j} + S_{1j} = 1.03.$
$S_{0j} + \alpha c_{1j} = 0.15 + (0.4)(2) = 0.95 < 1.$
For $j = 2: \ I_{0j} = \{1,2\}, I_{1,j} = \{3,4\}, c_{0j} = c_{1j} = 2.$
$S_{0j} = 0.10 + 0.10 = 0.20, S_{1j} = 0.40 + 0.40 = 0.80, S_{0j} + S_{1j} = 1.$
$S_{0j} + \alpha c_{1j} = 0.20 + (0.4)(2) = 1.$
For $j = 3: \ I_{0j} = \{2,3\}, I_{1,j} = \{1,4\}, c_{0j} = c_{1j} = 2.$
$S_{0j} = 0.08 + 0.12 = 0.20, S_{1j} = 0.40 + 0.40 = 0.80, S_{0j} + S_{1j} = 1.$
$S_{0j} + \alpha c_{1j} = 0.20 + (0.4)(2) = 1.$
For $j = 4: \ I_{0j} = \{1,4\}, I_{1j} = \{2,3\}, c_{0j} = c_{1j} = 2.$
$S_{0j} = 0.15 + 0.05 = 0.20, S_{1j} = 0.40 + 0.40 = 0.80, S_{0j} + S_{1j} = 1.$
$S_{0j} + \alpha c_{1j} = 0.20 + (0.4)(2) = 1.$
For $j = 5: \ I_{0j} = \{3,4\}, I_{1,j} = \{1,2\}, c_{0j} = c_{1j} = 2.$
$S_{0j} = 0.05 + 0.05 = 0.10, S_{1j} = 0.40 + 0.40 = 0.80, S_{0j} + S_{1j} = 0.9 < 1.$
$S_{0j} + \alpha c_{1j} = 0.1 + (0.4)(2) = 0.9 < 1.$
$\frac{1-(S_{0j}+S_{1j})}{c_{0j}} = \frac{1-0.95}{2} = 0.025 < \alpha - \max_{i \in I_{0j}}\{u_{(t)ij}\} = 0.4 - 0.05 = 0.35.$
Therefore the requirements of Method 2 are satisfied. Now we will construct columns of matrix Z.
For $j = 1: \ S_{0j} + S_{1j} = 1.03 > 1$, therefore $z_{31} = 0.05$ and $z_{41} = 0.10$.
In order to obtain z_{11} and z_{21} we need to use Algorithm 4.

Step 1. $I^1 := \{1,2\}, S = 1.03 - 1 = 0.03.$
Step 2. $D = \frac{0.03}{2}$, $I^2 = \{2\}$.
Step 3. Because $I^2 = \{2\}$, we get $z_{21} = \alpha = 0.4$ and $S = 0.03 - (0.4 - 0.4) = 0.03$, $I^1 := I^1 - I^2 = \{1\}$.
Step 2. $D = \frac{0.03}{1}$ and $I^1 = \emptyset$.
Step 3. $z_{11} = 0.48 - 0.03 = 0.45.$
For $j = 2: \ S_{0j} + S_{1j} = 1$, therefore $z_{12} = 0.10, z_{22} = 0.10, z_{32} = 0.40$ and $z_{42} = 0.40$.
Analogously for $j = 3$ and $j = 4$.
For $j = 5: \ S_{0j} + S_{1j} = 0.9 < 1$, therefore
$z_{15} = 0.40, z_{25} = 0.40$ and $z_{35} = 0.05 + \frac{1-0.9}{2} = 0.10, \ z_{45} = 0.05 + \frac{1-0.9}{2} = 0.10.$

The resulting partition is given by matrix

$$Z = \begin{pmatrix} 0.45 & 0.10 & 0.40 & 0.15 & .40 \\ 0.40 & 0.10 & 0.08 & 0.40 & .40 \\ 0.05 & 0.40 & 0.12 & 0.40 & .10 \\ 0.10 & 0.40 & 0.40 & 0.05 & .10 \end{pmatrix}.$$

It is obvious that we can find a lower bound and an upper bound of a collection of α-equivalent fuzzy k-partitions only if the requirements of Method 1 and Method 2 are satisfied. There are collections of α-equivalent fuzzy k-partitions where only a lower bound exists, only an upper bound exists, or neither lower nor upper bound exists.

Example 14
a) Consider fuzzy partitions U_1, U_2 from Example 11 and fuzzy partition V_1 given by matrix

$$V_1 = \begin{pmatrix} 0.48 & 0.10 & 0.54 & 0.15 & 0.45 \\ 0.42 & 0.10 & 0.01 & 0.40 & 0.55 \\ 0.05 & 0.40 & 0.05 & 0.40 & 0.00 \\ 0.05 & 0.40 & 0.40 & 0.05 & 0.00 \end{pmatrix}.$$

It is easy to verify that $U_1^\alpha = U_2^\alpha = V_1^\alpha$ for $\alpha = 0.4$. The α-upper bound of collection $\{U_1, U_2, V_1\}$ for $\alpha = 0.4$ is partition $Z \in P_{f40}$:

$$Z = \begin{pmatrix} 0.45 & 0.10 & 0.40 & 0.15 & .40 \\ 0.40 & 0.10 & 0.08 & 0.40 & .40 \\ 0.05 & 0.40 & 0.12 & 0.40 & .10 \\ 0.10 & 0.40 & 0.40 & 0.05 & .10 \end{pmatrix}.$$

However, the 0.4-lower bound of $\{U_1, U_2, V_1\}$ does not exist, because for $j = 5$ we get $S_{1j} = 0.5 + 0.55 = 1.05$, which means that the requirement $S_{1j} \leq 1$ of Method 1 is not satisfied.

b) Consider fuzzy partition U_3 from Example 11 and fuzzy partitions $V_2, V_3 \in P_{f4o}$ given by matrices

$$V_2 = \begin{pmatrix} 0.50 & 0.10 & 0.40 & 0.15 & .40 \\ 0.40 & 0.05 & 0.10 & 0.42 & .50 \\ 0.00 & 0.45 & 0.10 & 0.43 & .05 \\ 0.10 & 0.40 & 0.40 & 0.00 & .05 \end{pmatrix} \text{ and } V_3 = \begin{pmatrix} 0.50 & 0.10 & 0.45 & 0.15 & 0.50 \\ 0.45 & 0.00 & 0.13 & 0.45 & 0.40 \\ 0.00 & 0.40 & 0.02 & 0.40 & 0.05 \\ 0.05 & 0.50 & 0.40 & 0.00 & 0.05 \end{pmatrix}.$$

It is easy to verify that $U_3^\alpha = V_2^\alpha = V_3^\alpha$ for $\alpha = 0.4$. The α-lower bound of collection

$\{U_3, V_2, V_3\}$ for $\alpha = 0.4$ is partition $W \in P_{f40}$:

$$W = \begin{pmatrix} 0.50 & 0.05 & 0.555 & 0.12 & .50 \\ 0.45 & 0.00 & 0.010 & 0.45 & .50 \\ 0.00 & 0.45 & 0.020 & 0.43 & .00 \\ 0.05 & 0.50 & 0.415 & 0.00 & .00 \end{pmatrix}.$$

However, the 0.4-upper bound of $\{U_3, V_2, V_3\}$ does not exist, because for $j = 3$ we get: $S_{0j} + \alpha.c_{1j} = 0.23 + (0.4)(2) = 1.03$, which violates the requirement of Method 2, that $S_{0j} + \alpha.c_{1j} \leq 1$.

c) Consider partitions V_1, V_2 and V_3 from part a) and part b) of this example. $\{V_1, V_2, V_3\}$ is a collection of 0.4-equivalent fuzzy k-partitions. However, neither the the lower bound nor the upper bound of this collection exists with respect to α-sharpness, for $\alpha = 0.4$.

Method 1 can be viewed as a method of combination (aggregation) of fuzzy k-partitions with respect to minimal α-sharpness, while Method 2 can be viewed as a method of aggregation of fuzzy k-partitions with respect to maximal α-sharpness.

5 Conclusion

In this paper we have presented some techniques for comparison and combination of fuzzy k-partitions. There is clearly much to be done: further studies and generalization of measures of α-sharpness and measures of m-importance, aggregation of fuzzy partitions which are not α-equivalent, development of new relations of partial (or complete) order on P_{fko}. A challenging problem is to introduce some meaningful connectives on P_{fko} similar to t-norm and t-conorm on $\mathcal{F}(X)$. Some of these topics will be investigated in our further research.

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Appendix 1

We need to show that the partition W created by Method 1 is a fuzzy k-partition and that $W \prec_\alpha U_t$ for all $t \in N_m$.

It is clear from the construction of w_{ij} that $w_{ij} \geq 0$ for all $(ij) \in N_k \times N_n$. We will prove that $\sum_i w_{ij} = 1$ for all $j \in N_n$.

a) If $S_{0j} + S_{1j} \leq 1$ then:

$$\begin{aligned}\sum_i w_{ij} &= \sum_{i \in I_{0j}} w_{ij} + \sum_{i \in I_{1j}} w_{ij} = S_{0j} + \sum_{i \in I_{1j}} (\max_t\{u_{(t)ij}\} + \frac{1-(S_{0j}+S_{1j})}{c_{1j}}) \\ &= S_{0j} + \sum_{i \in I_{1j}} \max_t\{u_{(t)ij}\} + \sum_{i \in I_{1j}} \frac{1-(S_{0j}+S_{1j})}{c_{1j}} \\ &= S_{0j} + S_{1j} + c_{1j}\frac{1-(S_{0j}+S_{1j})}{c_{1j}} = 1.\end{aligned}$$

b) If $S_{0j} + S_{1j} > 1$ then:

Let card $I^1 = 0$ in the r-th iteration of Algorithm 3, $r \in \{1, 2, \ldots\}$. Then $I^{1(r)}$ is an empty set and $I^* = \cup_{p=1}^{r-1} I^{1(p)}$. It is obvious that $I_{0j} = I^* \cup I^{0(r)}$. According to Algorithm 3 we have that $w_{ij} = 0$ for all $i \in I^*$ and for all $i \in I^{0(r)}$ we have that $w_{ij} = \min_t\{u_{(t)ij}\} - \frac{(S_{0j}+S_{1j}-1)-\sum_{i\in I^*}\min_t\{u_{(t)ij}\}}{\text{card } I^{0(r)}}$. Therefore

$$\begin{aligned}\sum_i w_{ij} &= \sum_{i \in I_{1j}} w_{ij} + \sum_{i \in I_{0j}} w_{ij} = \sum_{i \in I_{1j}} w_{ij} + \sum_{i \in I^*} w_{ij} + \sum_{i \in I^{0(r)}} w_{ij} = \sum_{i \in I_{1j}} \max_t\{u_{(t)ij}\} \\ &+ \sum_{i \in I^{0(r)}} (\min_t\{u_{(t)ij}\} - \frac{(S_{0j}+S_{1j}-1) - \sum_{i\in I^*}\min_t\{u_{(t)ij}\}}{\text{card } I^{0(r)}}) \\ &= S_{1j} + \sum_{i \in I^{0(r)}} \min_t\{u_{(t)ij}\} - \text{ card } I^{0(r)} \frac{(S_{0j}+S_{1j}-1) - \sum_{i\in I^*}\min_t\{u_{(t)ij}\}}{\text{card } I^{0(r)}} \\ &= S_{1j} + \sum_{i \in I^{0(r)}} \min_t\{u_{(t)ij}\} - S_{0j} - S_{1j} + 1 + \sum_{i \in I^*} \min_t\{u_{(t)ij}\} \\ &= \sum_{i \in I^{0(r)}} \min_t\{u_{(t)ij}\} + \sum_{i \in I^*} \min_t\{u_{(t)ij}\} - S_{0j} + 1 \\ &= \sum_{i \in I_{0j}} \min_t\{u_{(t)ij}\} - S_{0j} + 1 = S_{0j} - S_{0j} + 1 = 1.\end{aligned}$$

We will prove that $W \prec_\alpha U$ and $W \prec_\alpha V$.

a) If $S_{0j} + S_{1j} \leq 1$ then:

For $i \in I_{0j}$ we have:

$w_{ij} = \min_t\{u_{(t)ij}\}$, therefore $w_{ij} \leq u_{(t)ij}$ for all $u_{(t)ij} < \alpha$ and all $t \in N_m$.

For $i \in I_{1j}$ we have:

$w_{ij} = \max_t\{u_{(t)ij}\} + \frac{1-(S_{0j}+S_{1j})}{c_{1j}}$. Because $\frac{1-(S_{0j}+S_{1j})}{c_{1j}} \geq 0$ we have that $w_{ij} \geq u_{(t)ij}$ for

all $u_{(t)ij} \geq \alpha$ and all $t \in N_m$.
Because $\sum_i w_{ij} = 1$ and $w_{ij} \geq 0$ for all $(ij) \in N_k \times N_n$, each coefficient w_{ij} defined by (4.6) is at most 1.

b) If $S_{0j} + S_{1j} > 1$ then:
For $i \in I_{0j}$ we have:
$w_{ij} \leq \min_t\{u_{(t)ij}\}$ therefore $w_{ij} \leq u_{(t)ij}$ for all $u_{(t)ij} < \alpha$ and all $t \in N_m$.
For $i \in I_{1j}$ we have:
$w_{ij} = \max_t\{u_{(t)ij}\}$, therefore $w_{ij} \geq u_{(t)ij}$ for all $u_{(t)ij} \geq \alpha$ and all $t \in N_m$.

Now we need to show that $\sum_i \sum_j (w_{ij} - \underline{s}_{ij})^2$ is minimal.
a) If $S_{0j} + S_{1j} = 1$, then $w_{ij} = \underline{s}_{ij}$, therefore $\sum_i (w_{ij} - \underline{s}_{ij})^2 = 0$.
b) If $S_{0j} + S_{1j} < 1$ then for $i \in I_{0j}: \; w_{ij} = \underline{s}_{ij}$ and for $i \in I_{1j}: \; w_{ij} = \underline{s}_{ij} + \delta_{ij}$.
Therefore $\sum_i (w_{ij} - \underline{s}_{ij})^2 = \sum_{i \in I_{0j}} (w_{ij} - \underline{s}_{ij})^2 + \sum_{i \in I_{1j}} (w_{ij} - \underline{s}_{ij})^2 = \sum_{i \in I_{1j}} \delta_{ij}^2$,
with the condition that $\sum_{i \in I_{1j}} \delta_{ij} = 1 - (S_{0j} + S_{1j}) = \Omega$.
We will use the method of Lagrange multipliers to prove that $\sum_{i \in I_{1j}} \delta_{ij}^2$ is minimal if $\delta_{ij} = \frac{\Omega}{c_{1j}}$ for all $i \in I_{1j}$.

Proof:
Let $F(\lambda, \delta_{ij}) = \sum_{i \in I_{1j}} \delta_{ij}^2 - \lambda(\sum_{i \in I_{ij}} \delta_{ij} - \Omega)$. Then

$$\frac{\partial F}{\partial \delta_{ij}} = 2\delta_{ij} - \lambda = 0, \tag{5.1}$$

$$\frac{\partial F}{\partial \lambda} = -\sum_{i \in I_{1j}} \delta_{ij} + \Omega = 0. \tag{5.2}$$

From (5.1) we get that

$$\delta_{ij} = \frac{\lambda}{2}. \tag{5.3}$$

Using (5.3) in (5.2) we get the solution for λ,

$$\lambda = \frac{2\Omega}{c_{1j}}. \tag{5.4}$$

Substituting (5.4) for λ in (5.3) we get

$$\delta_{ij} = \frac{\Omega}{c_{1j}} = \frac{1 - (S_{0j} + S_{1j})}{c_{1j}}. \tag{5.5}$$

c) The proof of case when $S_{0j} + S_{1j} > 1$ is analogous to the proof of case b).

Appendix 2

We need to show that the partition Z created by Method 2 is a fuzzy k-partition. It is clear from the construction of z_{ij} that $z_{ij} \geq 0$ for all $(ij) \in N_k \times N_n$. We will prove that $\sum_i z_{ij} = 1$ for all $j \in N_n$.

a) If $S_{0j} + S_{1j} \leq 1$ then:

$$\begin{aligned}
\sum_i z_{ij} &= \sum_{i \in I_{1j}} z_{ij} + \sum_{i \in I_{0j}} z_{ij} = S_{1j} + \sum_{i \in I_{0j}} (\max_t \{u_{(t)ij}\} + \frac{1-(S_{0j}+S_{1j})}{c_{0j}}) \\
&= S_{1j} + \sum_{i \in I_{0j}} \max_t \{u_{(t)ij}\} + \sum_{i \in I_{0j}} \frac{1-(S_{0j}+S_{1j})}{c_{0j}} \\
&= S_{1j} + S_{0j} + c_{0j} \frac{1-(S_{0j}+S_{1j})}{c_{0j}} = 1.
\end{aligned}$$

b) If $S_{0j} + S_{1j} > 1$ then:

Let card $I^2 = 0$ in the r-th iteration of Algorithm 4, $r \in \{1, 2, \ldots\}$. Then $I^{2(r)}$ is an empty set and $I^* = \cup_{p=1}^{r-1} I^{2(p)}$. It is obvious that $I_{1j} = I^* \cup I^{1(r)}$. According to Algorithm 4 we have that $z_{ij} = \alpha$ for all $i \in I^*$ and for all $i \in I^{1(r)}$ we have $z_{ij} = \min_t \{u_{(t)ij}\} - \frac{(S_{0j}+S_{1j}-1) - \sum_{i \in I^*} (\min_t \{u_{(t)ij}\} - \alpha)}{\text{card } I^{1(r)}}$. Therefore

$$\begin{aligned}
\sum_i z_{ij} &= \sum_{i \in I_{0j}} z_{ij} + \sum_{i \in I_{1j}} z_{ij} = \sum_{i \in I_{0j}} \max_t \{u_{(t)ij}\} + \sum_{i \in I^*} \alpha \\
&+ \sum_{i \in I^{1(r)}} (\min_t \{u_{(t)ij}\} - \frac{(S_{0j}+S_{1j}-1) - \sum_{i \in I^*} (\min_t \{u_{(t)ij}\} - \alpha)}{\text{card } I^{1(r)}}) \\
&= S_{0j} + \text{card } I^*.\alpha + \sum_{i \in I^{1(r)}} \min_t \{u_{(t)ij}\} \\
&- \text{card } I^{1(r)} \frac{(S_{0j}+S_{1j}-1) - \sum_{i \in I^*} (\min_t \{u_{(t)ij}\} - \alpha)}{\text{card } I^{1(r)}} \\
&= S_{0j} + \text{card } I^*.\alpha + \sum_{i \in I^{1(r)}} \min_t \{u_{(t)ij}\} - S_{0j} - S_{1j} + 1 \\
&+ \sum_{i \in I^*} \min_t \{u_{(t)ij}\} - \text{card } I^*.\alpha = \sum_{i \in I_{1j}} \min_t \{u_{(t)ij}\} - S_{1j} + 1 \\
&= S_{1j} - S_{1j} + 1 = 1.
\end{aligned}$$

The proof that $U_t \prec_\alpha Z$ for all $t \in N_m$ and that $\sum_i \sum_j (z_{ij} - \overline{s}_{ij})^2$ is minimal is analogous to the proof in Appendix 1.