

# Selection Model in Functional Linear Regression Models for Scalar Response

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## Abstract

The so-called Functional Linear Regression model consists in explaining a scalar response by a regressor which is a random function observed on a compact subset of  $\mathbb{R}$ : in this context, the “parameter” of linear model is a function of the weights.

In order to estimate this functional coefficient some estimators such as Functional Principal Component Regression Estimator, Smooth Principal Component Regression Estimator, Penalized B-Splines Estimator, have been introduced in literature. We focus our attention on the Functional Principal Component Regression Estimator and in particular on the connected dimensionality problem.

Our aim is to apply and compare some different selection methods, which have been proposed in the classical regression field. These methods are illustrated and compared by the means of simulations.

## 1 Introduction

In many statistical frameworks we have to face problems involving data that are curves, or more exactly, collections of discrete observations effected on curves: these data are classified as “functionals”. For example we can think of longitudinal data analyzed in meteorology, in medicine and biology, in economy and/or finance, or the spectrometric data observed in chemometrics, or the digitalized signals analyzed in the signals analysis.

Many techniques for the analysis of data with functional nature which allows the variability exploration in samples of curves and let the research of common structures among the curves or groups of them have been introduced; for an exhaustive presentation of these methods, see the monograph of Ramsay and Silverman, 1997.

A very interesting application consists in the possibility of constructing regression models through which we describe the relation between a real variable and an explanatory variable having functional nature. Such a model is termed as functional regression model. Several examples can be found in literature: Hastie and Mallows

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(1993) propose some application to quantitative chemistry, Marx and Eilers (1996) illustrate a phonemes classification by means of the log-spectra of a sequence of spoken syllables, Ramsay and Silverman (1997) describe the estimation of the total annual precipitation in some Canadian weather station from the temperature curves measured during the year, Cardot *et al.* (1999b) propose a forecasting model in order to explain winter wheat yield as a linear function of the duration of the crop and climatic variations, Ferraty and Vieu (2002) illustrate the prevision of fat content in some meet samples from the respective spectrometric curves.

Formally in the functional regression model the link between a real random variable (r.v.)  $Y$  and a functional random variable  $X = \{X(t), t \in \mathcal{T}\}$ , where  $\mathcal{T}$  is the interval of observation, is described by the relation

$$Y = \mu + \Psi(X) + \varepsilon \quad (1.1)$$

where  $\mu$  is a real constant,  $\Psi$  is a real operator and  $\varepsilon$  is a zero mean random variable with finite variance, which we assume non-correlated with  $X$ .

In this paper we consider the special case where  $X$  is a r.v. mapping in  $H = L^2(\mathcal{T})$ , the separable Hilbert space of square integrable functions defined on  $\mathcal{T} \subset \mathbb{R}$ ,  $\mathcal{T}$  compact. In this framework  $\Psi$  is a real-valued linear continuous operator. By the Riesz Representation Theorem, there exists a unique function  $\psi \in H$  such that

$$\Psi(x) = \int_{\mathcal{T}} \psi(t)x(t)dt, \quad x \in H, \quad (1.2)$$

Hence model (1.1) may be rewritten as

$$Y = \mu + \int_{\mathcal{T}} \psi(t)X(t)dt + \varepsilon. \quad (1.3)$$

Model (1.3) is a generalization of the linear regression model to the case of an infinite number of regressors. Many authors dwell upon the estimation of the functional coefficient  $\psi$  and/or the operator  $\Psi$ : in some cases criteria are based on functional bases expansion of the  $\psi$  function. We mention for example the Penalized Splines Estimators studied in Marx and Eilers (1999) and in Cardot *et al.* (1999b and 2002), and the Fourier Basis Expansion Estimators treated in Ramsay and Silverman (1997).

In this work we use the estimation method introduced by Cardot *et al.* (1999a) and based on Principal Component Regression (PCR). This technique consists in a mean square estimation of regression models whose regressors are the  $k$  principal components of  $X$ , obtained from the  $k$  eigenfunctions associated to the  $k$  largest eigenvalues of the covariance operator of  $X$ . An important problem to solve is the determination of the parameter  $k$  that represents the dimension of the subspace on which we project the observations.

In order to introduce the notations we will use in the following and to clarify the model and the estimator, we devote Section 2 to a quick presentation of the functional linear model and to the discussion of the above mentioned Functional PCR Estimator.

In Section 3 we dwell upon to some selection criteria that have been proposed in the classical regression tools. These methods are applied to the problem of the optimal dimension choice and they are compared through simulation in Section 4.

## 2 Functional Linear Regression Model and Functional PCR Estimator

Let  $H = L^2_{\mathcal{T}}$  be the Hilbert space of square integrable functions defined on a compact set  $\mathcal{T} \subset \mathbb{R}$ , provided of the inner product  $\langle \cdot, \cdot \rangle$  defined for all  $f, g \in H$  as  $\langle f, g \rangle = \int_{\mathcal{T}} f(t)g(t)dt$ .

We consider a r.v.  $(X, Y)$  defined on the probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and taking values into the measurable space  $(\mathbb{R} \times H, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_H)$ . We assume moreover that  $Y \in L^2_{\mathbb{R}}$  and  $X \in L^2_H$ .

Let  $m(\cdot)$  be the conditional mean of  $Y$  given  $X$ , *i.e.*

$$m(x) = \mathbb{E}(Y | \{X(t) = x(t), t \in \mathcal{T}\}), \quad x \in H.$$

When it is linear we can write

$$m(x) = \mu + \langle \psi, x \rangle = \mu + \Psi(x), \quad x \in H.$$

In general, since we cannot assume a linear specification for  $m(\cdot)$  we consider an approximation of it, which may be obtained as a solution of the least square problem

$$(\mu, \psi) = \arg \min_{(\alpha, \xi) \in \mathbb{R} \times H} \mathbb{E}[(Y - \alpha - \langle \xi, X \rangle)^2].$$

Existence and unicity of the solution are discussed in Cardot *et al.* (2003).

The covariance operator of  $X$  and the cross-covariance operator between  $X$  and  $Y$  are defined respectively as:

$$\begin{aligned} \Gamma(x) &= \mathbb{E}[\langle X - \mathbb{E}(X), x \rangle (X - \mathbb{E}(X))], & x \in H \\ \Delta(x) &= \mathbb{E}[\langle X - \mathbb{E}(X), x \rangle (Y - \mathbb{E}(Y))], & x \in H \end{aligned}$$

These two operators are related by  $\Delta = \Psi\Gamma$ . Since the operator  $\Gamma$  is nuclear (Dauxois *et al.*, 1982), its inverse is not bounded when  $H$  has infinite dimension: to solve this drawback the idea proposed at first by Bosq in the ARH models (Bosq, 1991 and 2000) and used in Cardot *et al.* (1999a), consists of projecting  $X$ 's observations on a finite  $k$ -dimensional subspace  $H_k$  of  $H$ .

The basic tool is the Karhunen-Loeve decomposition of a second order random function:

$$X = \sum_{j=1}^{\infty} \langle X, v_j \rangle v_j = \sum_{j=1}^{\infty} \tilde{X}_j v_j$$

where  $v_j$  are the orthonormal eigenfunctions of  $\Gamma$ , sorted according to the decreasing sequence of associated eigenvalues  $\lambda_j$ . We note that this decomposition, truncated at order  $k < \infty$ , gives the best linear representation of  $X$  in the sense that it maximizes the explained variance: in practice it is a Principal Component Analysis, which we call "Functional". In this context the real r.v.  $\tilde{X}_j$  plays the role of the  $j$ -th principal component of  $X$ .

As proven by Cardot *et al.* (2003), the operator  $\Psi$  has the following approximation

$$\Psi_{\text{PCR}}(\cdot) = \sum_{j=1}^k \frac{\Delta v_j}{\lambda_j} \langle v_j, \cdot \rangle$$

and equivalently we have

$$\psi_{\text{PCR}} = \sum_{j=1}^k \frac{\Delta v_j}{\lambda_j} v_j.$$

In order to estimate  $\Psi$  (or equivalently  $\psi$ ) we consider an i.i.d. sample  $(X_i, Y_i)_{i=1, \dots, n}$  drawn from the r.v.  $(X, Y)$ .

Define

$$\begin{aligned} \Gamma_n(x) &= \frac{1}{n} \sum_{i=1}^n \langle (X_i - \bar{X}), x \rangle (X_i - \bar{X}), & x \in H \\ \Delta_n(x) &= \frac{1}{n} \sum_{i=1}^n \langle (X_i - \bar{X}), x \rangle (Y_i - \bar{Y}), & x \in H \end{aligned}$$

the empirical versions of  $\Gamma$  and  $\Delta$ , where  $\bar{X} = \{n^{-1} \sum_{i=1}^n X_i(t), t \in \mathcal{T}\}$  and  $\bar{Y} = n^{-1} \sum_{i=1}^n Y_i$ .

Let  $\{\hat{v}_j, \hat{\lambda}_j\}_{j=1, 2, \dots}$  be the eigenelements of  $\Gamma_n$ , with  $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \dots \geq \hat{\lambda}_n = 0$ , and let  $\mathbb{P}_k$  be the orthogonal projection on  $\hat{H}_k$ , the space spanned by  $\{\hat{v}_j\}_{j=1, \dots, k}$ . The PCR estimators of  $\Psi$  and  $\psi$  are given respectively by

$$\hat{\Psi}_{\text{PCR}}(\cdot) = \Delta_n \mathbb{P}_k (\mathbb{P}_k \Gamma_n \mathbb{P}_k)^{-1} = \sum_{j=1}^k \frac{\Delta_n \hat{v}_j}{\hat{\lambda}_j} \langle \hat{v}_j, \cdot \rangle$$

and

$$\hat{\psi}_{\text{PCR}} = \sum_{j=1}^k \frac{\Delta_n \hat{v}_j}{\hat{\lambda}_j} \hat{v}_j.$$

Thus this method consists in an ordinary least square regression of the response r.v.  $Y$  on the  $k$  variables  $\tilde{X}_j$ , the first  $k$  principal components of  $X$ . Under suitable conditions on  $k$  and on the spectrum of  $\Gamma$  and  $\Gamma_n$ , Cardot *et al.* (1999a) have proven the almost sure (and in probability) convergence for the PCR estimator.

As Cardot *et al.* (2003) point out, the technique gives a rough estimation of the function  $\psi$ , also when sample size  $n$  is large. In order to solve this problem the authors introduce a second step in the estimation procedure, consisting in a smoothing of  $\hat{\psi}$  by means of a B-spline approximation.

### 3 Model selection criteria

As we have seen in the previous section, in order to construct PCR estimator, the functional linear regression model is approximated in an optimal way by using a truncated Functional Principal Component Analysis. So we may write:

$$\begin{aligned} Y &= \mu + \Psi(X) + \varepsilon \cong \mu + \Psi \left( \sum_{j=1}^k \tilde{X}_j v_j \right) + \varepsilon \\ &= \mu + \sum_{j=1}^k \tilde{X}_j \Psi(v_j) + \varepsilon = \mu + \sum_{j=1}^k \tilde{X}_j \beta_j + \varepsilon, \end{aligned}$$

where the eigenfunctions  $v_j$  may be estimated by  $\widehat{v}_j$ . This is a regression model whose explanatory variables are the first  $k$  principal components of  $X$ , ordered according to the sequence of eigenvalues  $\lambda_j$ , that are the variances of principal components (*i.e.*  $\text{Var}(\widetilde{X}_j) = \lambda_j$ ).

In order to construct the estimator, we need to determine of the optimal dimension  $k$ : this is a problem of variables choice in a linear regression model, a very important subject as the wide literature on the topic shows (for some analysis and discussions about selection methods, see, for example, the monographs of Miller, 1990, McQuarrie and Tsai, 1998, and Burnham and Anderson, 2002). In this work we focus our attention on some choice criteria widely used in the empirical analysis.

We consider the problem of model selection into a family of potential candidates, indexed by  $h = 1, 2, \dots$ . Let  $\theta_h = (\mu, \beta_1, \beta_2, \dots, \beta_{k_h}, \sigma^2)$  be the vector of parameters of the  $h$ -th regression model, and  $\Theta_h \subset \mathbb{R}^{k_h+2}$  the parameter space for  $\theta_h$ . Denote by  $d_h = k_h + 2$  the dimension of  $\Theta_h$  and by  $p_h = k_h + 1$  the number of coefficients in the candidate regression model.

We recall that in the PCR Estimation framework it is often suggested to order the principal component according to the magnitude of correlation with the explained variable  $Y$ , since the principal components with largest variability (*i.e.* associated to the largest eigenvalues) are not necessarily the most explanatory.

In our case, we can not adopt this approach since it could give some non convergent estimators. In fact, one of the hypotheses ensuring the a.s. (and in probability) convergence of PCR estimators is that sequences of eigenvalues  $\{\lambda_j\}_{j=1,2,\dots}$  and  $\{\widehat{\lambda}_j\}_{j=1,2,\dots}$  decrease (Cardot *et al.*, 1999a), and so principal components must be ranged consequently.

So, for  $h = 1$ , the parameter is  $\theta_1 = (\mu, \sigma^2)$ ; for  $h = 2$  we introduce the first principal component  $\widetilde{X}_1$  and we have to estimate  $\theta_2 = (\mu, \beta_1, \sigma^2)$ . When  $h = 3$  we use the first two principal components  $\widetilde{X}_1$  and  $\widetilde{X}_2$ , and so  $\theta_3 = (\mu, \beta_1, \beta_2, \sigma^2)$ . The generic  $\theta_h$  is constructed according to this rule.

In order to select the “good” regression model, some authors suggest to choose model  $h$  which minimizes the quantity

$$-2 \log(L(h)) + pen(h) \quad (3.1)$$

where  $\log(L(h))$  is the log-likelihood computed for  $h$ -model and  $pen(h)$  is a penalty term connected to the dimension of models. Under gaussian conditions for the noise variable ( $\varepsilon \sim \mathcal{N}(0, \sigma^2)$  with  $\sigma^2 < \infty$ ) criteria (3.1) may be expressed by the operative formula

$$n \log(\widehat{\sigma}_h^2) + \delta(h) \quad (3.2)$$

where  $\widehat{\sigma}_h^2 = n^{-1} \sum_{i=1}^n (Y_i - \widehat{Y}_i^{(h)})^2$  is the ML estimator of  $\sigma^2$ , and  $\widehat{Y}_i^{(h)}$  is the estimation of  $Y_i$  obtained by using the  $h$ -model.

Of this nature are the criteria introduced by Schwartz (1978) called Bayesian Information Criterion (BIC), in which  $\delta(h) = d_h \log n$  and the Akaike’s Information Criterion - AIC, (Akaike, 1973). In this work, we do not use the original AIC with  $\delta(h) = 2d_h$ , instead we make use of the “corrected” version AICc proposed by Hurvich and Tsai (1989) in which  $\delta(h) = n(n+p_h)/(n-p_h-2)$  with  $n > p_h+2$ . This

choice is motivated by the results of Cavanaugh (1997) which proposes an unified vision of AIC and AICc.

Similar to the previous methods are the criteria proposed by Hannan and Quinn (1979) with  $\delta(h) = 2d_h \log(\log n)$ , and the so-called Rice's  $T$  (Rice, 1984), where  $\delta(h) = n \log(1 - 2p_h/n)$  with  $p_h < n/2$ . The latter has been originally introduced in the context of choice of bandwidth in the kernel regression framework and here it has been adapted to the parametric case.

An other criterion that we consider is the one proposed by Shibata (1981) for which the optimal  $p_h$  minimizes the statistic

$$S(h) = \hat{\sigma}_h^2(n + 2p_h).$$

This criterion has been proposed for some gaussian model that can be specified by an infinite number of parameters and so it may be used validly in our context.

The last approach we have considered is the classical Cross Validation consisting of setting aside of data, usually one observation at a time, and making predictions of these data by using the remaining ones. Let  $\hat{Y}_i^{(h,-i)}$  be the prediction of  $Y_i$  based on the other  $n - 1$  observations. We choose a  $p_h$  minimizing

$$\frac{1}{n} \sum_{i=1}^n (Y_i - \hat{Y}_i^{(h,-i)})^2.$$

Since we treat linear models, it may be proved that it is equivalent to choose the number of coefficients  $p_h$  which minimizes the following *PRESS* statistic (Allen, 1974):

$$PRESS(h) = \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{Y}_i^{(h)})^2 / (1 - \Pi_i)^2$$

where  $\Pi_i$  is the  $i$ -th term of the principal diagonal of the projection matrix onto the  $p_h$ -dimensional subspace of  $\mathbb{R}^n$ .

## 4 Numerical examples

This section is devoted to numerical simulations of the performances of the criteria presented in Section 3: we make comparisons by using some different functional coefficients with various sample sizes.

In practice we compare the relative empirical *MISE* calculated for each criterion, defined as follows:

$$RMISE^{(c)} = \frac{1}{M} \sum_{m=1}^M \frac{\int_{\mathcal{I}} (\psi(t) - \hat{\psi}_m^{(c)}(t))^2 dt}{\int_{\mathcal{I}} (\psi(t))^2 dt}$$

where  $M$  is the number of simulations in each study case and  $\hat{\psi}_m^{(c)}(t)$  is the estimation of the functional coefficient obtained at the  $m$ -th simulation applying the  $c$ -th selection method.

In our study we base simulations on the following operative conditions:

1.  $X$  is a standard brownian motion defined on the set  $\mathcal{T} = [0, 1]$  and whose trajectories are discretized in 100 equispaced points,
2. sample sizes are  $n = 50, 100, 200, 500$  and 1000,
3. parameters of functional linear regression models are
  - (a)  $\mu = 2$  for all models,
  - (b) functional coefficients defined on  $[0, 1]$  as follows
    - i.  $\psi_1(t) = \sin(.5\pi t) + .5 \sin(1.5\pi t) + .25 \sin(2.5\pi t)$ ,
    - ii.  $\psi_2(t) = \sin(4\pi t)$ ,
    - iii.  $\psi_3(t) = \sin(6\pi t)$ ,
    - iv.  $\psi_4(t) = (\sin(2\pi t^2))^3$ ,
    - v.  $\psi_5(t) = 1 - 48t + 218t^2 - 315t^3 + 145t^4$ ,
    - vi.  $\psi_6(t) = 0.3 \exp(-64(t - .25)^2) + 0.7 \exp(-256(t - .75)^2)$ .
4. For each model we introduce noise variables which have gaussian, uniform and centered gamma distribution. Their variability is controlled by the following signal-to-noise ratio:

$$snr = \frac{\sigma_{\Psi(X)}^2}{\sigma_{\varepsilon}^2 + \sigma_{\Psi(X)}^2},$$

where  $\sigma_{\Psi(X)}^2 = \text{Var}(\Psi(X))$  and  $\sigma_{\varepsilon}^2 = \text{Var}(\varepsilon)$ . In simulations we use  $snr = 65\%$ ,  $85\%$  and  $95\%$ , which allow respectively to some error variables with large variance ( $\sigma_{\varepsilon}^2/\sigma_{\Psi(X)}^2 = 0.54$ ), moderate variance ( $\sigma_{\varepsilon}^2/\sigma_{\Psi(X)}^2 = 0.18$ ) and small variance ( $\sigma_{\varepsilon}^2/\sigma_{\Psi(X)}^2 = 0.05$ ).

5. The number of simulations for each setting of experimental factors is  $M = 500$ .

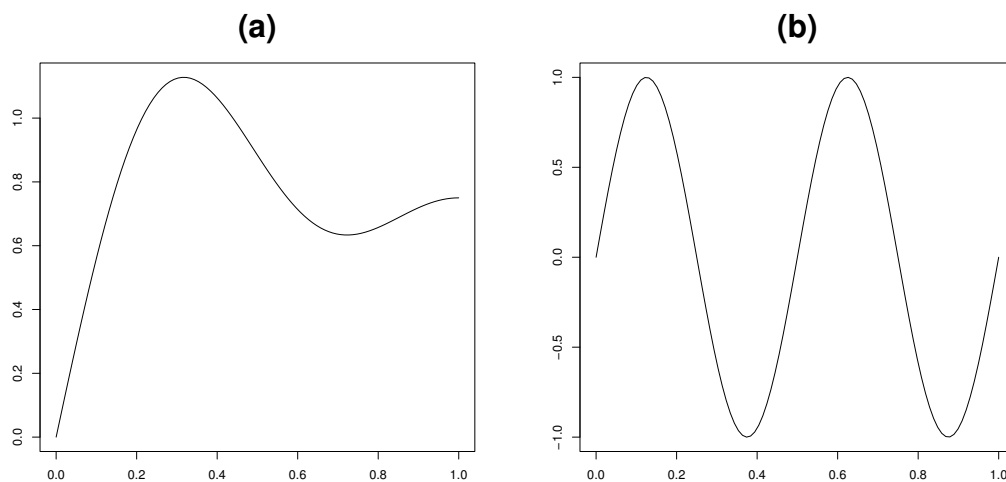
Since the eigenelements of the covariance operator for a standard brownian motion on  $[0, 1]$  are (see Ash and Gardner, 1975):

$$\lambda_j = \frac{1}{(j - 0.5)^2 \pi^2}, \quad v_j(t) = \sqrt{2} \sin((j - 0.5)\pi t), \quad t \in [0, 1] \quad j = 1, 2, \dots,$$

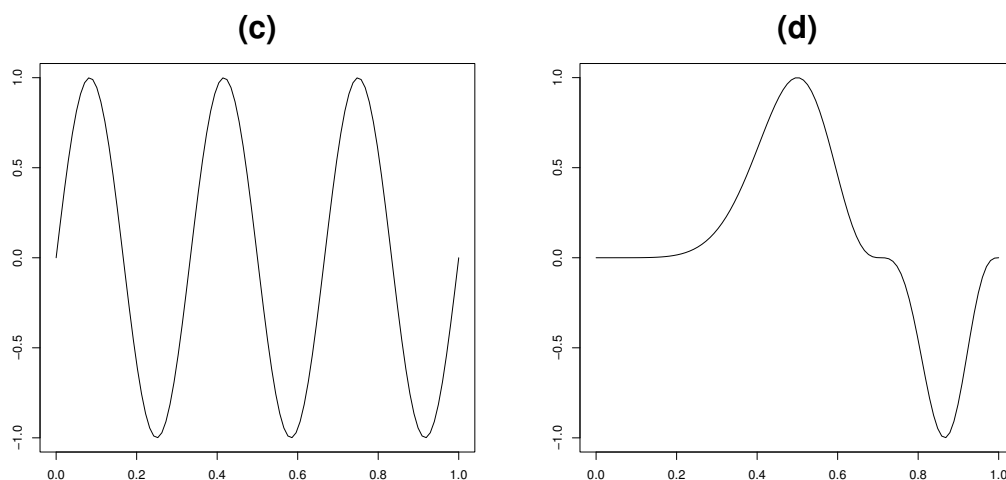
we may observe that the functional coefficient  $\psi_1$  is built matching the first three eigenfunctions  $v_j$  in an opportune way.

$\psi_2$  and  $\psi_3$  are periodic functions which can not be decomposed into a finite sum of eigenfunctions, the second one having a large amount of fine structure.  $\psi_4$  is more general.  $\psi_5$  is a function which combines a less fine structure with a trend.  $\psi_6$  has different degrees of curvature for different values of  $t$ . The shape of the functional coefficients are shown in Figures 1, 2, and 3.

Tables 1, 2, 3, 4, 5, and 6 give *RMISE* values computed for the different functional coefficients  $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$  and  $\psi_6$ , under various error distribution hypotheses and using the selection criteria BIC (Schwartz), AICc (Hurvich and Tsai), HQ (Hannan and Quinn), S (Shibata), T (Rice) and PRESS (Allen).



**Figure 1:** Functional coefficients for simulations: (a)  $\psi_1$ , and (b)  $\psi_2$ .

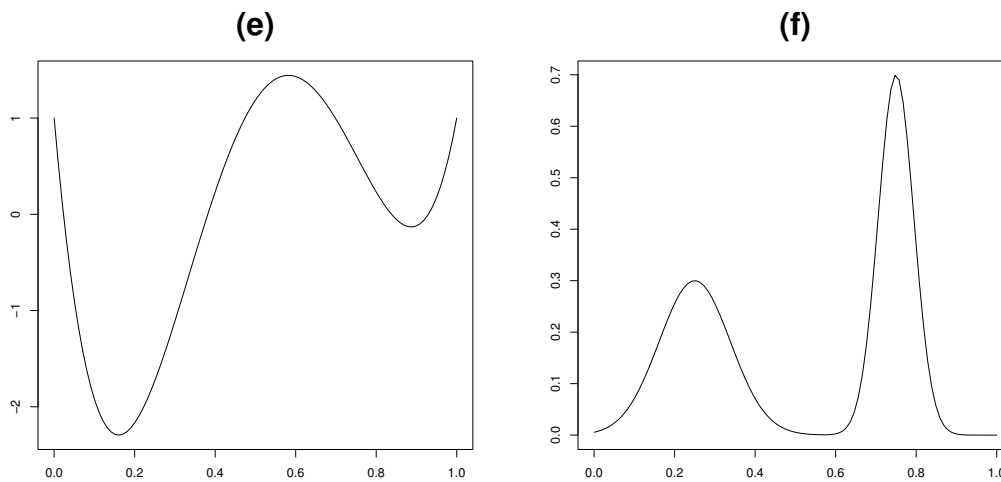


**Figure 2:** Functional coefficients for simulations: (c)  $\psi_3$  and (d)  $\psi_4$ .

As general observation, we can note that the properties of the selectors differ for the various cases, but certain patterns emerge anyway, and in particular they are linked to the sample sizes, the signal-to-noise ratio ( $snr$  in the follows) and to the distribution of the noise variables.

It is clear that when the sample size increases, the performances of selectors improve, since estimations are more accurate. In very small samples, with  $n = 50$ , all approaches are unacceptable also when  $snr$  is large ( $snr = 0.95$ ). However, in some cases, we obtain very large  $RMISE$  values also when  $n \geq 500$  (for example, see results in Table 6). Besides in general the criteria seem not to be equivalent also for very large values of  $n$  ( $n = 1000$ ). Analogously, selectors have different behaviors





**Figure 3:** Functional coefficients for simulations: (e)  $\psi_5$  and (f)  $\psi_6$ .

when  $snr$  change: for each sample size,  $RMISE$  is reduced substantially when  $snr$  increases.

Another general remark is that for any case, each selector tends to reproduce the same behavior, for any error distribution. Thus, comments that may be done for the gaussian cases can be repeated changing the distributions of the error variable, both when the distribution law is symmetric (uniform case) and both when it is asymmetric (centered gamma case). This may justify the use of methods defined in gaussian context in more general cases.

In the following we propose a short discussion for each estimated model, in order to supply some practical indication for the use of selection methods.

When we estimate the functional coefficient  $\psi_1$ , BIC gives good results, constantly for any distribution error and  $snr$  value. For example, with  $snr = 0.85$  and  $n \geq 100$   $RMISE$  is smaller than 10%. The Shibata criterion is similar and sometimes better than BIC when the sample size is large (for  $n = 500$  and  $snr = 0.85$ , we have  $RMISE = 2.6\%$  for the gaussian case and  $RMISE = 2.7\%$  for the other cases). The other criteria are not optimal, and in particular the one of Hannan and Quinn behaves very badly. Anyway all these criteria seem to have the same behavior for  $n \geq 500$ , although they give worse performances compared to BIC and S: for example, for  $n \geq 500$  and  $snr = 0.85$ , they supply  $RMISE$  that are 4 – 6 times bigger than the best ones.

Cases of estimate of periodic functions  $\psi_2$  and  $\psi_3$  are rather similar with respect to the behavior of the selectors: for  $n \geq 200$  the Shibata's criterion is good with  $RMISE \leq 6\%$  for  $snr = 0.85$ , and  $RMISE \leq 3.5\%$  when  $snr = 0.95$ , consistently for any noise distribution. Hannan and Quinn criterion perform usually in a poor way even if, for very large sample size and  $snr \geq 0.85$ , it seems equivalent to the other criteria.

The case of coefficient  $\psi_4$  confirms that the Shibata's criterion is a good method of selection for large sample under different error conditions and  $snr$  values: in fact,

**Table 1:**  $\psi_1(t) = \sin((1 - .5)\pi t) + .5 \sin((2 - .5)\pi t) + .25 \sin((3 - .5)\pi t)$ .

|                      | n    | BIC    | AICc   | HQ      | S      | T      | PRESS  |
|----------------------|------|--------|--------|---------|--------|--------|--------|
| Gaussian Error       |      |        |        |         |        |        |        |
| snr = 0.65           | 50   | 0.3051 | 0.7514 | 9.1796  | 0.7544 | 0.8137 | 1.4909 |
|                      | 100  | 0.1791 | 0.9235 | 2.8973  | 0.3775 | 1.0898 | 1.7138 |
|                      | 200  | 0.1030 | 0.6501 | 1.2088  | 0.1988 | 0.6820 | 0.8321 |
|                      | 500  | 0.0569 | 0.2901 | 0.3690  | 0.0703 | 0.2909 | 0.3097 |
|                      | 1000 | 0.0421 | 0.1525 | 0.1726  | 0.0359 | 0.1525 | 0.1556 |
| snr = 0.85           | 50   | 0.1795 | 0.4083 | 3.5345  | 0.5232 | 0.4416 | 0.7730 |
|                      | 100  | 0.0749 | 0.3531 | 1.1757  | 0.1482 | 0.3814 | 0.5520 |
|                      | 200  | 0.0537 | 0.2341 | 0.4274  | 0.0811 | 0.2502 | 0.3106 |
|                      | 500  | 0.0282 | 0.0938 | 0.1154  | 0.0256 | 0.0940 | 0.0952 |
|                      | 1000 | 0.0123 | 0.0674 | 0.0731  | 0.0121 | 0.0675 | 0.0678 |
| snr = 0.95           | 50   | 0.0640 | 0.1201 | 1.3176  | 0.1789 | 0.1432 | 0.3127 |
|                      | 100  | 0.0406 | 0.1073 | 0.3701  | 0.0733 | 0.1180 | 0.1844 |
|                      | 200  | 0.0207 | 0.0815 | 0.1386  | 0.0239 | 0.0863 | 0.1037 |
|                      | 500  | 0.0055 | 0.0328 | 0.0395  | 0.0074 | 0.0339 | 0.0370 |
|                      | 1000 | 0.0025 | 0.0163 | 0.0187  | 0.0036 | 0.0163 | 0.0173 |
| Centered Gamma Error |      |        |        |         |        |        |        |
| snr = 0.65           | 50   | 0.3006 | 0.8081 | 10.6547 | 1.3322 | 0.8793 | 2.4219 |
|                      | 100  | 0.1840 | 0.9484 | 3.1345  | 0.3849 | 1.0845 | 1.2942 |
|                      | 200  | 0.1105 | 0.5858 | 1.0809  | 0.2064 | 0.6042 | 0.6842 |
|                      | 500  | 0.0540 | 0.3359 | 0.4534  | 0.0624 | 0.3358 | 0.3882 |
|                      | 1000 | 0.0419 | 0.1649 | 0.1814  | 0.0427 | 0.1648 | 0.1757 |
| snr = 0.85           | 50   | 0.2612 | 0.3865 | 3.3807  | 0.4651 | 0.4296 | 0.8452 |
|                      | 100  | 0.0860 | 0.4034 | 1.0718  | 0.1857 | 0.4318 | 0.5794 |
|                      | 200  | 0.0508 | 0.2632 | 0.4835  | 0.0937 | 0.2852 | 0.3110 |
|                      | 500  | 0.0292 | 0.0854 | 0.1103  | 0.0273 | 0.0854 | 0.0910 |
|                      | 1000 | 0.0121 | 0.0594 | 0.0656  | 0.0123 | 0.0594 | 0.0622 |
| snr = 0.95           | 50   | 0.0689 | 0.1211 | 1.1191  | 0.1709 | 0.1335 | 0.2593 |
|                      | 100  | 0.0408 | 0.1325 | 0.4054  | 0.0762 | 0.1332 | 0.1661 |
|                      | 200  | 0.0225 | 0.0793 | 0.1234  | 0.0230 | 0.0819 | 0.0895 |
|                      | 500  | 0.0058 | 0.0289 | 0.0327  | 0.0076 | 0.0290 | 0.0302 |
|                      | 1000 | 0.0026 | 0.0203 | 0.0229  | 0.0045 | 0.0203 | 0.0208 |
| Uniform Error        |      |        |        |         |        |        |        |
| snr = 0.65           | 50   | 0.3377 | 0.9470 | 10.5057 | 0.9449 | 1.0260 | 1.5423 |
|                      | 100  | 0.2226 | 0.6962 | 2.9884  | 0.3444 | 0.7351 | 1.2375 |
|                      | 200  | 0.1145 | 0.5396 | 0.9475  | 0.1971 | 0.5889 | 0.6844 |
|                      | 500  | 0.0570 | 0.2505 | 0.3224  | 0.0649 | 0.2575 | 0.2776 |
|                      | 1000 | 0.0424 | 0.1556 | 0.1639  | 0.0382 | 0.1559 | 0.1539 |
| snr = 0.85           | 50   | 0.2275 | 0.4183 | 3.8087  | 0.7557 | 0.4927 | 1.1520 |
|                      | 100  | 0.0907 | 0.3585 | 1.1977  | 0.1609 | 0.4235 | 0.6175 |
|                      | 200  | 0.0652 | 0.2285 | 0.3593  | 0.0766 | 0.2321 | 0.2903 |
|                      | 500  | 0.0306 | 0.0995 | 0.1311  | 0.0266 | 0.1039 | 0.1097 |
|                      | 1000 | 0.0130 | 0.0684 | 0.0768  | 0.0112 | 0.0685 | 0.0660 |
| snr = 0.95           | 50   | 0.0875 | 0.1443 | 1.1126  | 0.1896 | 0.1574 | 0.3003 |
|                      | 100  | 0.0383 | 0.1147 | 0.3480  | 0.0631 | 0.1213 | 0.1516 |
|                      | 200  | 0.0194 | 0.0751 | 0.1255  | 0.0237 | 0.0754 | 0.0843 |
|                      | 500  | 0.0050 | 0.0376 | 0.0482  | 0.0076 | 0.0377 | 0.0378 |
|                      | 1000 | 0.0024 | 0.0176 | 0.0204  | 0.0038 | 0.0190 | 0.0186 |

**Table 2:**  $\psi_2(t) = \sin(4\pi t)$ .

|                      | n    | BIC    | AICc   | HQ     | S      | T      | PRESS  |
|----------------------|------|--------|--------|--------|--------|--------|--------|
| Gaussian Error       |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.3957 | 0.4316 | 1.0495 | 0.5045 | 0.4331 | 0.5687 |
|                      | 100  | 0.1775 | 0.2500 | 0.4171 | 0.2157 | 0.2574 | 0.3030 |
|                      | 200  | 0.0995 | 0.1338 | 0.1718 | 0.1058 | 0.1349 | 0.1412 |
|                      | 500  | 0.0521 | 0.0685 | 0.0733 | 0.0500 | 0.0685 | 0.0697 |
|                      | 1000 | 0.0336 | 0.0435 | 0.0451 | 0.0326 | 0.0436 | 0.0439 |
| snr=0.85             | 50   | 0.2247 | 0.2252 | 0.4487 | 0.2655 | 0.2264 | 0.2828 |
|                      | 100  | 0.0982 | 0.1217 | 0.1794 | 0.1098 | 0.1222 | 0.1371 |
|                      | 200  | 0.0552 | 0.0683 | 0.0774 | 0.0562 | 0.0683 | 0.0714 |
|                      | 500  | 0.0301 | 0.0329 | 0.0340 | 0.0280 | 0.0329 | 0.0334 |
|                      | 1000 | 0.0206 | 0.0195 | 0.0199 | 0.0184 | 0.0195 | 0.0196 |
| snr=0.95             | 50   | 0.1537 | 0.1539 | 0.2192 | 0.1685 | 0.1544 | 0.1726 |
|                      | 100  | 0.0633 | 0.0683 | 0.0867 | 0.0649 | 0.0689 | 0.0734 |
|                      | 200  | 0.0338 | 0.0364 | 0.0395 | 0.0327 | 0.0366 | 0.0372 |
|                      | 500  | 0.0177 | 0.0169 | 0.0172 | 0.0165 | 0.0169 | 0.0171 |
|                      | 1000 | 0.0115 | 0.0103 | 0.0104 | 0.0103 | 0.0103 | 0.0103 |
| Centered Gamma Error |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.3598 | 0.3644 | 1.0799 | 0.4365 | 0.3696 | 0.4901 |
|                      | 100  | 0.1731 | 0.2419 | 0.4215 | 0.2132 | 0.2486 | 0.2854 |
|                      | 200  | 0.0990 | 0.1452 | 0.1971 | 0.1054 | 0.1477 | 0.1571 |
|                      | 500  | 0.0532 | 0.0687 | 0.0753 | 0.0513 | 0.0695 | 0.0704 |
|                      | 1000 | 0.0333 | 0.0397 | 0.0414 | 0.0315 | 0.0401 | 0.0401 |
| snr=0.85             | 50   | 0.2266 | 0.2332 | 0.4776 | 0.2848 | 0.2343 | 0.2965 |
|                      | 100  | 0.0995 | 0.1250 | 0.1901 | 0.1087 | 0.1276 | 0.1395 |
|                      | 200  | 0.0564 | 0.0746 | 0.0898 | 0.0571 | 0.0761 | 0.0788 |
|                      | 500  | 0.0304 | 0.0333 | 0.0346 | 0.0286 | 0.0333 | 0.0334 |
|                      | 1000 | 0.0210 | 0.0199 | 0.0203 | 0.0189 | 0.0199 | 0.0200 |
| snr=0.95             | 50   | 0.1504 | 0.1522 | 0.2183 | 0.1644 | 0.1508 | 0.1665 |
|                      | 100  | 0.0612 | 0.0652 | 0.0841 | 0.0625 | 0.0656 | 0.0702 |
|                      | 200  | 0.0333 | 0.0374 | 0.0409 | 0.0336 | 0.0374 | 0.0384 |
|                      | 500  | 0.0175 | 0.0174 | 0.0178 | 0.0162 | 0.0174 | 0.0175 |
|                      | 1000 | 0.0115 | 0.0103 | 0.0104 | 0.0102 | 0.0103 | 0.0102 |
| Uniform Error        |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.3639 | 0.3723 | 1.0344 | 0.4607 | 0.3784 | 0.5307 |
|                      | 100  | 0.1686 | 0.2311 | 0.3984 | 0.1857 | 0.2381 | 0.2785 |
|                      | 200  | 0.0991 | 0.1325 | 0.1749 | 0.0965 | 0.1367 | 0.1457 |
|                      | 500  | 0.0518 | 0.0645 | 0.0716 | 0.0479 | 0.0644 | 0.0681 |
|                      | 1000 | 0.0342 | 0.0395 | 0.0404 | 0.0317 | 0.0396 | 0.0398 |
| snr=0.85             | 50   | 0.2198 | 0.2285 | 0.4528 | 0.2666 | 0.2272 | 0.2814 |
|                      | 100  | 0.0983 | 0.1158 | 0.1669 | 0.1089 | 0.1192 | 0.1267 |
|                      | 200  | 0.0553 | 0.0680 | 0.0833 | 0.0564 | 0.0687 | 0.0734 |
|                      | 500  | 0.0292 | 0.0345 | 0.0364 | 0.0276 | 0.0347 | 0.0352 |
|                      | 1000 | 0.0202 | 0.0193 | 0.0196 | 0.0179 | 0.0194 | 0.0195 |
| snr=0.95             | 50   | 0.1562 | 0.1606 | 0.2374 | 0.1728 | 0.1599 | 0.1822 |
|                      | 100  | 0.0645 | 0.0697 | 0.0869 | 0.0668 | 0.0702 | 0.0757 |
|                      | 200  | 0.0336 | 0.0367 | 0.0405 | 0.0326 | 0.0370 | 0.0378 |
|                      | 500  | 0.0176 | 0.0171 | 0.0176 | 0.0162 | 0.0171 | 0.0172 |
|                      | 1000 | 0.0114 | 0.0101 | 0.0102 | 0.0102 | 0.0101 | 0.0101 |

**Table 3:**  $\psi_3(t) = \sin(6\pi t)$ .

|                      | n    | BIC    | AICc   | HQ     | S      | T      | PRESS  |
|----------------------|------|--------|--------|--------|--------|--------|--------|
| Gaussian Error       |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.5418 | 0.4906 | 0.6664 | 0.5245 | 0.4851 | 0.5209 |
|                      | 100  | 0.2254 | 0.2449 | 0.3089 | 0.2306 | 0.2475 | 0.2592 |
|                      | 200  | 0.1187 | 0.1341 | 0.1469 | 0.1170 | 0.1359 | 0.1377 |
|                      | 500  | 0.0618 | 0.0628 | 0.0647 | 0.0588 | 0.0628 | 0.0631 |
|                      | 1000 | 0.0393 | 0.0380 | 0.0385 | 0.0361 | 0.0381 | 0.0382 |
| snr=0.85             | 50   | 0.2865 | 0.2823 | 0.3258 | 0.2869 | 0.2803 | 0.2902 |
|                      | 100  | 0.1340 | 0.1374 | 0.1518 | 0.1377 | 0.1370 | 0.1408 |
|                      | 200  | 0.0706 | 0.0716 | 0.0738 | 0.0692 | 0.0717 | 0.0722 |
|                      | 500  | 0.0362 | 0.0317 | 0.0320 | 0.0329 | 0.0317 | 0.0318 |
|                      | 1000 | 0.0230 | 0.0192 | 0.0191 | 0.0199 | 0.0192 | 0.0191 |
| snr=0.95             | 50   | 0.1983 | 0.1960 | 0.1920 | 0.1933 | 0.1961 | 0.1948 |
|                      | 100  | 0.0857 | 0.0832 | 0.0847 | 0.0831 | 0.0830 | 0.0835 |
|                      | 200  | 0.0455 | 0.0414 | 0.0412 | 0.0422 | 0.0414 | 0.0414 |
|                      | 500  | 0.0218 | 0.0182 | 0.0180 | 0.0196 | 0.0182 | 0.0181 |
|                      | 1000 | 0.0135 | 0.0107 | 0.0106 | 0.0117 | 0.0107 | 0.0107 |
| Centered Gamma Error |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.5378 | 0.4795 | 0.6981 | 0.5355 | 0.4754 | 0.5243 |
|                      | 100  | 0.2253 | 0.2470 | 0.3085 | 0.2351 | 0.2479 | 0.2571 |
|                      | 200  | 0.1212 | 0.1366 | 0.1517 | 0.1200 | 0.1369 | 0.1394 |
|                      | 500  | 0.0603 | 0.0644 | 0.0661 | 0.0565 | 0.0644 | 0.0644 |
|                      | 1000 | 0.0396 | 0.0369 | 0.0370 | 0.0363 | 0.0369 | 0.0371 |
| snr=0.85             | 50   | 0.2972 | 0.2861 | 0.3339 | 0.3008 | 0.2845 | 0.2986 |
|                      | 100  | 0.1304 | 0.1373 | 0.1518 | 0.1329 | 0.1377 | 0.1400 |
|                      | 200  | 0.0731 | 0.0755 | 0.0779 | 0.0723 | 0.0756 | 0.0763 |
|                      | 500  | 0.0355 | 0.0323 | 0.0324 | 0.0327 | 0.0323 | 0.0324 |
|                      | 1000 | 0.0231 | 0.0192 | 0.0191 | 0.0202 | 0.0192 | 0.0190 |
| snr=0.95             | 50   | 0.1951 | 0.1899 | 0.1863 | 0.1868 | 0.1894 | 0.1869 |
|                      | 100  | 0.0828 | 0.0816 | 0.0824 | 0.0819 | 0.0817 | 0.0817 |
|                      | 200  | 0.0455 | 0.0416 | 0.0413 | 0.0426 | 0.0416 | 0.0414 |
|                      | 500  | 0.0209 | 0.0181 | 0.0180 | 0.0192 | 0.0181 | 0.0181 |
|                      | 1000 | 0.0136 | 0.0109 | 0.0108 | 0.0118 | 0.0109 | 0.0109 |
| Uniform Error        |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.5301 | 0.4673 | 0.6754 | 0.5156 | 0.4661 | 0.5073 |
|                      | 100  | 0.2219 | 0.2430 | 0.3024 | 0.2282 | 0.2445 | 0.2557 |
|                      | 200  | 0.1195 | 0.1320 | 0.1469 | 0.1177 | 0.1320 | 0.1357 |
|                      | 500  | 0.0596 | 0.0611 | 0.0629 | 0.0563 | 0.0612 | 0.0614 |
|                      | 1000 | 0.0389 | 0.0367 | 0.0369 | 0.0346 | 0.0368 | 0.0368 |
| snr=0.85             | 50   | 0.2958 | 0.2890 | 0.3281 | 0.2913 | 0.2853 | 0.2906 |
|                      | 100  | 0.1287 | 0.1299 | 0.1467 | 0.1284 | 0.1300 | 0.1350 |
|                      | 200  | 0.0720 | 0.0716 | 0.0747 | 0.0690 | 0.0716 | 0.0725 |
|                      | 500  | 0.0355 | 0.0341 | 0.0343 | 0.0332 | 0.0342 | 0.0341 |
|                      | 1000 | 0.0233 | 0.0191 | 0.0190 | 0.0203 | 0.0191 | 0.0191 |
| snr=0.95             | 50   | 0.1990 | 0.1953 | 0.1922 | 0.1940 | 0.1951 | 0.1956 |
|                      | 100  | 0.0851 | 0.0833 | 0.0839 | 0.0834 | 0.0832 | 0.0829 |
|                      | 200  | 0.0455 | 0.0424 | 0.0420 | 0.0435 | 0.0424 | 0.0422 |
|                      | 500  | 0.0218 | 0.0188 | 0.0185 | 0.0198 | 0.0187 | 0.0186 |
|                      | 1000 | 0.0134 | 0.0107 | 0.0106 | 0.0116 | 0.0107 | 0.0107 |

**Table 4:**  $\psi_4(t) = (\sin(2\pi t^2))^3$ .

|                      | n    | BIC    | AICc   | HQ     | S      | T      | PRESS  |
|----------------------|------|--------|--------|--------|--------|--------|--------|
| Gaussian Error       |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.3840 | 0.4621 | 1.5207 | 0.5364 | 0.4690 | 0.6364 |
|                      | 100  | 0.2710 | 0.3556 | 0.6475 | 0.3111 | 0.3572 | 0.4110 |
|                      | 200  | 0.2079 | 0.2529 | 0.3326 | 0.2116 | 0.2566 | 0.2722 |
|                      | 500  | 0.1430 | 0.1564 | 0.1602 | 0.1380 | 0.1573 | 0.1568 |
|                      | 1000 | 0.1188 | 0.0892 | 0.0896 | 0.1018 | 0.0892 | 0.0895 |
| snr=0.85             | 50   | 0.2667 | 0.2868 | 0.6534 | 0.3209 | 0.2869 | 0.3643 |
|                      | 100  | 0.1805 | 0.2142 | 0.3077 | 0.1899 | 0.2157 | 0.2308 |
|                      | 200  | 0.1376 | 0.1396 | 0.1535 | 0.1333 | 0.1400 | 0.1419 |
|                      | 500  | 0.0966 | 0.0599 | 0.0614 | 0.0635 | 0.0599 | 0.0603 |
|                      | 1000 | 0.0480 | 0.0321 | 0.0327 | 0.0300 | 0.0322 | 0.0323 |
| snr=0.95             | 50   | 0.1897 | 0.1943 | 0.3146 | 0.2090 | 0.1947 | 0.2227 |
|                      | 100  | 0.1154 | 0.1104 | 0.1334 | 0.1093 | 0.1110 | 0.1165 |
|                      | 200  | 0.0677 | 0.0558 | 0.0614 | 0.0541 | 0.0557 | 0.0569 |
|                      | 500  | 0.0237 | 0.0248 | 0.0251 | 0.0221 | 0.0248 | 0.0249 |
|                      | 1000 | 0.0142 | 0.0145 | 0.0146 | 0.0137 | 0.0145 | 0.0145 |
| Centered Gamma Error |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.3766 | 0.4026 | 1.5415 | 0.5222 | 0.4289 | 0.6315 |
|                      | 100  | 0.2716 | 0.3786 | 0.6567 | 0.3138 | 0.3840 | 0.4268 |
|                      | 200  | 0.2029 | 0.2602 | 0.3607 | 0.2169 | 0.2644 | 0.2735 |
|                      | 500  | 0.1429 | 0.1551 | 0.1585 | 0.1391 | 0.1564 | 0.1564 |
|                      | 1000 | 0.1172 | 0.0870 | 0.0893 | 0.0966 | 0.0872 | 0.0881 |
| snr=0.85             | 50   | 0.2563 | 0.2878 | 0.7081 | 0.3336 | 0.2910 | 0.3614 |
|                      | 100  | 0.1802 | 0.2236 | 0.3113 | 0.1979 | 0.2249 | 0.2408 |
|                      | 200  | 0.1393 | 0.1492 | 0.1654 | 0.1349 | 0.1482 | 0.1524 |
|                      | 500  | 0.0924 | 0.0621 | 0.0635 | 0.0660 | 0.0620 | 0.0621 |
|                      | 1000 | 0.0497 | 0.0335 | 0.0340 | 0.0301 | 0.0335 | 0.0338 |
| snr=0.95             | 50   | 0.1872 | 0.1909 | 0.2986 | 0.2153 | 0.1906 | 0.2198 |
|                      | 100  | 0.1178 | 0.1124 | 0.1342 | 0.1134 | 0.1131 | 0.1166 |
|                      | 200  | 0.0692 | 0.0592 | 0.0653 | 0.0585 | 0.0595 | 0.0620 |
|                      | 500  | 0.0252 | 0.0251 | 0.0258 | 0.0229 | 0.0252 | 0.0254 |
|                      | 1000 | 0.0142 | 0.0148 | 0.0151 | 0.0134 | 0.0149 | 0.0148 |
| Uniform Error        |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.3663 | 0.4251 | 1.5492 | 0.5513 | 0.4458 | 0.6496 |
|                      | 100  | 0.2628 | 0.3704 | 0.6321 | 0.2954 | 0.3733 | 0.4414 |
|                      | 200  | 0.2011 | 0.2599 | 0.3238 | 0.2032 | 0.2627 | 0.2751 |
|                      | 500  | 0.1412 | 0.1562 | 0.1654 | 0.1401 | 0.1566 | 0.1593 |
|                      | 1000 | 0.1178 | 0.0885 | 0.0904 | 0.0984 | 0.0885 | 0.0885 |
| snr=0.85             | 50   | 0.2662 | 0.2963 | 0.6546 | 0.3366 | 0.3020 | 0.3617 |
|                      | 100  | 0.1812 | 0.1994 | 0.2821 | 0.1933 | 0.2027 | 0.2240 |
|                      | 200  | 0.1360 | 0.1413 | 0.1596 | 0.1316 | 0.1416 | 0.1461 |
|                      | 500  | 0.0959 | 0.0641 | 0.0663 | 0.0668 | 0.0642 | 0.0647 |
|                      | 1000 | 0.0451 | 0.0325 | 0.0333 | 0.0294 | 0.0325 | 0.0327 |
| snr=0.95             | 50   | 0.1968 | 0.2015 | 0.3279 | 0.2272 | 0.2006 | 0.2400 |
|                      | 100  | 0.1168 | 0.1112 | 0.1318 | 0.1109 | 0.1126 | 0.1189 |
|                      | 200  | 0.0708 | 0.0606 | 0.0639 | 0.0578 | 0.0607 | 0.0620 |
|                      | 500  | 0.0244 | 0.0250 | 0.0259 | 0.0239 | 0.0251 | 0.0253 |
|                      | 1000 | 0.0143 | 0.0141 | 0.0142 | 0.0133 | 0.0141 | 0.0142 |

**Table 5:**  $\psi_5(t) = 1 - 48t + 218t^2 - 315t^3 + 145t^4$ .

|                      | n    | BIC    | AICc   | HQ     | S      | T      | PRESS  |
|----------------------|------|--------|--------|--------|--------|--------|--------|
| Gaussian Error       |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.2635 | 0.3612 | 1.7660 | 0.4526 | 0.3779 | 0.6061 |
|                      | 100  | 0.1633 | 0.3104 | 0.6456 | 0.2429 | 0.3228 | 0.3661 |
|                      | 200  | 0.1020 | 0.1879 | 0.2475 | 0.1138 | 0.1888 | 0.2003 |
|                      | 500  | 0.0611 | 0.1056 | 0.1113 | 0.0668 | 0.1061 | 0.1095 |
|                      | 1000 | 0.0510 | 0.0811 | 0.0849 | 0.0547 | 0.0812 | 0.0831 |
| snr=0.85             | 50   | 0.2000 | 0.2317 | 0.7109 | 0.2795 | 0.2374 | 0.3230 |
|                      | 100  | 0.0989 | 0.1441 | 0.2632 | 0.1148 | 0.1453 | 0.1699 |
|                      | 200  | 0.0681 | 0.1033 | 0.1303 | 0.0745 | 0.1039 | 0.1098 |
|                      | 500  | 0.0521 | 0.0632 | 0.0663 | 0.0540 | 0.0633 | 0.0643 |
|                      | 1000 | 0.0465 | 0.0503 | 0.0513 | 0.0459 | 0.0503 | 0.0504 |
| snr=0.95             | 50   | 0.1527 | 0.1694 | 0.3760 | 0.1963 | 0.1718 | 0.2174 |
|                      | 100  | 0.0781 | 0.0953 | 0.1465 | 0.0845 | 0.0985 | 0.1077 |
|                      | 200  | 0.0570 | 0.0684 | 0.0771 | 0.0585 | 0.0685 | 0.0696 |
|                      | 500  | 0.0450 | 0.0442 | 0.0455 | 0.0419 | 0.0443 | 0.0446 |
|                      | 1000 | 0.0368 | 0.0334 | 0.0334 | 0.0339 | 0.0334 | 0.0334 |
| Centered Gamma Error |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.2669 | 0.3244 | 1.8680 | 0.4057 | 0.3476 | 0.5814 |
|                      | 100  | 0.1617 | 0.3009 | 0.6513 | 0.2079 | 0.3297 | 0.3722 |
|                      | 200  | 0.0973 | 0.1885 | 0.2790 | 0.1116 | 0.1911 | 0.2058 |
|                      | 500  | 0.0612 | 0.1108 | 0.1280 | 0.0662 | 0.1124 | 0.1171 |
|                      | 1000 | 0.0513 | 0.0778 | 0.0829 | 0.0546 | 0.0781 | 0.0812 |
| snr=0.85             | 50   | 0.1884 | 0.2197 | 0.7900 | 0.2650 | 0.2270 | 0.3229 |
|                      | 100  | 0.1048 | 0.1585 | 0.3120 | 0.1316 | 0.1741 | 0.2032 |
|                      | 200  | 0.0693 | 0.1229 | 0.1493 | 0.0765 | 0.1239 | 0.1265 |
|                      | 500  | 0.0524 | 0.0656 | 0.0696 | 0.0544 | 0.0658 | 0.0668 |
|                      | 1000 | 0.0470 | 0.0514 | 0.0522 | 0.0463 | 0.0515 | 0.0514 |
| snr=0.95             | 50   | 0.1429 | 0.1576 | 0.3421 | 0.1767 | 0.1585 | 0.1851 |
|                      | 100  | 0.0748 | 0.0944 | 0.1358 | 0.0851 | 0.0951 | 0.1065 |
|                      | 200  | 0.0558 | 0.0695 | 0.0793 | 0.0569 | 0.0702 | 0.0733 |
|                      | 500  | 0.0444 | 0.0456 | 0.0462 | 0.0425 | 0.0457 | 0.0457 |
|                      | 1000 | 0.0373 | 0.0342 | 0.0342 | 0.0343 | 0.0342 | 0.0340 |
| Uniform Error        |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.2600 | 0.3515 | 1.7317 | 0.5167 | 0.3737 | 0.6430 |
|                      | 100  | 0.1579 | 0.2822 | 0.6103 | 0.2056 | 0.3015 | 0.3709 |
|                      | 200  | 0.1044 | 0.1875 | 0.2820 | 0.1149 | 0.1888 | 0.2081 |
|                      | 500  | 0.0602 | 0.0990 | 0.1118 | 0.0663 | 0.1003 | 0.1040 |
|                      | 1000 | 0.0509 | 0.0758 | 0.0780 | 0.0531 | 0.0759 | 0.0763 |
| snr=0.85             | 50   | 0.1786 | 0.2301 | 0.7175 | 0.2484 | 0.2331 | 0.3087 |
|                      | 100  | 0.1005 | 0.1443 | 0.2767 | 0.1217 | 0.1513 | 0.1756 |
|                      | 200  | 0.0678 | 0.1068 | 0.1348 | 0.0724 | 0.1081 | 0.1157 |
|                      | 500  | 0.0514 | 0.0657 | 0.0716 | 0.0523 | 0.0660 | 0.0667 |
|                      | 1000 | 0.0463 | 0.0510 | 0.0514 | 0.0455 | 0.0510 | 0.0512 |
| snr=0.95             | 50   | 0.1543 | 0.1725 | 0.3632 | 0.2001 | 0.1764 | 0.2178 |
|                      | 100  | 0.0790 | 0.0959 | 0.1454 | 0.0878 | 0.0964 | 0.1077 |
|                      | 200  | 0.0567 | 0.0692 | 0.0762 | 0.0581 | 0.0694 | 0.0716 |
|                      | 500  | 0.0444 | 0.0456 | 0.0467 | 0.0424 | 0.0456 | 0.0458 |
|                      | 1000 | 0.0373 | 0.0338 | 0.0338 | 0.0339 | 0.0338 | 0.0338 |

**Table 6:**  $\psi_6(t) = 0.3 \exp(-64(t - .25)^2) + 0.7 \exp(-256(t - .75)^2)$ .

|                      | n    | BIC    | AICc   | HQ     | S      | T      | PRESS  |
|----------------------|------|--------|--------|--------|--------|--------|--------|
| Gaussian Error       |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.7655 | 1.0836 | 5.0320 | 1.2000 | 1.1393 | 1.4901 |
|                      | 100  | 0.7020 | 1.1661 | 2.1737 | 0.8439 | 1.1947 | 1.3220 |
|                      | 200  | 0.6419 | 0.9985 | 1.3474 | 0.6253 | 1.0096 | 1.1309 |
|                      | 500  | 0.4290 | 0.5208 | 0.5500 | 0.3513 | 0.5253 | 0.5251 |
|                      | 1000 | 0.2467 | 0.3274 | 0.3438 | 0.2498 | 0.3320 | 0.3344 |
| snr=0.85             | 50   | 0.6656 | 0.7496 | 2.3707 | 0.9529 | 0.7609 | 1.1440 |
|                      | 100  | 0.5353 | 0.5880 | 0.9913 | 0.5120 | 0.6157 | 0.6555 |
|                      | 200  | 0.3437 | 0.4287 | 0.4876 | 0.3287 | 0.4314 | 0.4412 |
|                      | 500  | 0.2139 | 0.2635 | 0.2750 | 0.2236 | 0.2636 | 0.2655 |
|                      | 1000 | 0.1879 | 0.1823 | 0.1852 | 0.1768 | 0.1823 | 0.1830 |
| snr=0.95             | 50   | 0.4066 | 0.4205 | 0.8989 | 0.4842 | 0.4281 | 0.5326 |
|                      | 100  | 0.2700 | 0.3212 | 0.4777 | 0.2875 | 0.3248 | 0.3620 |
|                      | 200  | 0.2127 | 0.2297 | 0.2486 | 0.2155 | 0.2303 | 0.2363 |
|                      | 500  | 0.1655 | 0.1191 | 0.1200 | 0.1280 | 0.1191 | 0.1188 |
|                      | 1000 | 0.1077 | 0.0724 | 0.0724 | 0.0724 | 0.0724 | 0.0728 |
| Centered Gamma Error |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.7908 | 1.0304 | 5.6731 | 1.2359 | 1.1515 | 1.6671 |
|                      | 100  | 0.6666 | 1.0793 | 2.4654 | 0.8090 | 1.1604 | 1.3408 |
|                      | 200  | 0.6130 | 0.8390 | 1.1636 | 0.6194 | 0.8421 | 0.8765 |
|                      | 500  | 0.4360 | 0.5266 | 0.5903 | 0.3743 | 0.5341 | 0.5476 |
|                      | 1000 | 0.2490 | 0.3357 | 0.3567 | 0.2418 | 0.3358 | 0.3399 |
| snr=0.85             | 50   | 0.6737 | 0.7484 | 2.0863 | 0.8230 | 0.7552 | 0.9239 |
|                      | 100  | 0.5133 | 0.6437 | 1.1074 | 0.5813 | 0.6713 | 0.7006 |
|                      | 200  | 0.3592 | 0.4394 | 0.5298 | 0.3397 | 0.4519 | 0.4723 |
|                      | 500  | 0.2164 | 0.2632 | 0.2783 | 0.2259 | 0.2642 | 0.2655 |
|                      | 1000 | 0.1897 | 0.1772 | 0.1781 | 0.1813 | 0.1771 | 0.1770 |
| snr=0.95             | 50   | 0.4247 | 0.4192 | 0.8852 | 0.4747 | 0.4183 | 0.5121 |
|                      | 100  | 0.2769 | 0.3197 | 0.4313 | 0.2980 | 0.3248 | 0.3444 |
|                      | 200  | 0.2122 | 0.2249 | 0.2523 | 0.2103 | 0.2250 | 0.2278 |
|                      | 500  | 0.1667 | 0.1222 | 0.1243 | 0.1342 | 0.1226 | 0.1218 |
|                      | 1000 | 0.1042 | 0.0728 | 0.0729 | 0.0736 | 0.0727 | 0.0727 |
| Uniform Error        |      |        |        |        |        |        |        |
| snr=0.65             | 50   | 0.7189 | 1.1792 | 5.7416 | 1.4728 | 1.3589 | 1.9794 |
|                      | 100  | 0.6762 | 1.0807 | 2.1893 | 0.7817 | 1.0984 | 1.2609 |
|                      | 200  | 0.6208 | 0.8705 | 1.2099 | 0.6604 | 0.8729 | 0.9545 |
|                      | 500  | 0.4296 | 0.5109 | 0.5474 | 0.3322 | 0.5118 | 0.5154 |
|                      | 1000 | 0.2454 | 0.3321 | 0.3455 | 0.2397 | 0.3326 | 0.3379 |
| snr=0.85             | 50   | 0.6838 | 0.7620 | 2.2574 | 0.8946 | 0.7563 | 1.0579 |
|                      | 100  | 0.5345 | 0.5978 | 0.9791 | 0.5490 | 0.6134 | 0.6909 |
|                      | 200  | 0.3468 | 0.4223 | 0.5317 | 0.3359 | 0.4262 | 0.4417 |
|                      | 500  | 0.2115 | 0.2814 | 0.2986 | 0.2226 | 0.2813 | 0.2864 |
|                      | 1000 | 0.1889 | 0.1772 | 0.1792 | 0.1812 | 0.1773 | 0.1766 |
| snr=0.95             | 50   | 0.4234 | 0.4134 | 0.9229 | 0.4724 | 0.4174 | 0.5153 |
|                      | 100  | 0.2816 | 0.3218 | 0.4754 | 0.3027 | 0.3389 | 0.3634 |
|                      | 200  | 0.2104 | 0.2349 | 0.2577 | 0.2127 | 0.2378 | 0.2441 |
|                      | 500  | 0.1657 | 0.1247 | 0.1271 | 0.1297 | 0.1246 | 0.1254 |
|                      | 1000 | 0.1055 | 0.0773 | 0.0774 | 0.0771 | 0.0773 | 0.0771 |

for  $n \geq 200$ , the calculated  $RMISE$  is not larger than 14% (it reaches 3% for  $n = 1000$ ) when  $snr = 0.85$  and 6% when  $snr = 0.95$  (with  $RMISE = 1.3\%$  for  $n = 1000$ ). BIC gives the best results for small samples, even if  $RMISE$  values are rather high.

For the models whose coefficient is  $\psi_5$ , simulations indicate Shibata's criterion as a good one for  $n \geq 200$  whereas BIC is good for small samples. In fact when  $n \geq 100$ ,  $RMISE$  obtained using the Schwartz's method is not larger than 11% when  $snr = 0.85$  or  $0.95$ , and S performs similarly or better for  $n \geq 200$ . The HQ method is the worst one, even if all methods seem to be equivalents for  $n = 1000$  and  $snr = 0.95$ .

When we estimate the coefficient  $\psi_6$ , we obtain very bad performances in general: when  $n$  and  $snr$  are large all selectors perform poorly with  $RMISE$  not lower than 7 – 8%. For example, when  $snr = 0.85$  and  $n = 1000$ , the best criterions give  $RMISE$  around 18% (for example in the gaussian case the Shibata's criterion is the best with  $RMISE = 17.7\%$ ).

At this point, in the light of previous comments, we can give some general practical indications. It seems that for large samples ( $n \geq 500$ ) the Shibata's selector gives good results: in general, it supplies the best performances or at least near to the best ones, for any error distribution and signal-to-noise ratio.

When the sample size is small ( $n \leq 200$ ), BIC is performing often better than the others one, even if the  $RMISE$  is high. We may note that it gives better results than the corrected AIC, which should be the a priori suitable selector for small samples. The criterion that supplies the worst results is that of Hannan and Quinn: usually it performs very badly and in the best cases it gives  $RMISE$  at least of the order of the other methods.

For the others selection methods, it seems that Rice's T, PRESS and AIC have a quite similar behavior for large samples, but in general they have not homogeneous and good performances in the different analyzed cases. We note also that the PRESS criterion, which is very used in the applications and in the software implementations, is not optimal also with large samples.

## 5 Conclusions and perspectives

As we have already noticed in this paper, when we use the Principal Component Functional Regression Estimator in order to estimate the functional linear regression model, we can introduce many techniques to choose the number of Principal Components: as simulations show, it seems that some criteria are better than others, and in particular the Shibata's criterion gives always good performances for large samples. These results remain valid also when models are not gaussian.

The attainment of convergence results in  $L^2$ -norm is a gap which has to be filled: if we had these results we could verify if, using the above mentioned selection criteria, the estimator attains the optimal  $L^2$ -rate of convergence.

As we previously mentioned, the PCR estimator which we have used here is not the only one which needs the choice of a regularization parameter. In particular the suggested results should be read in the case in which estimation is based on Fourier



Basis Expansion: in this case the functions of the base are chosen a priori and there is the need to identify their number.

When we use the Penalized spline method, the problem does not lie in the choice of the number of the knots, but in the smoothing parameter which controls the roughness of the curves (see Marx and Eilers, 1996, for a discussion on this topic).

Another perspective is the possibility of extending the results obtained in the i.i.d. context to the case of dependent observations: the same kind of analysis presented here should be made in the time series framework using functional linear models. This approach should lead to use this tool in a dependent context: also in this case we need some robust selection criterion to face the dimensionality problem.

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