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A Markovian Model for Investment Analysis in Advertising

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Abstract

The present work deals with an application of statistics to different choices of investment in advertising. The distribution of market-shares of two telephone guide companies (G_A and G_B) has been analyzed and in particular the choices of investment made by lots of companies from 1997 to 2001 have been taken into account. Starting from the time series of their investments in G_A and G_B, we have tried to identify the probability for the companies to join the competitive group. The aim of the research is reached through the use of Markov chains to discover the various types of investing behaviour in the long period. Finally, through a stratification and a separate estimate of the transition probabilities, the main differences of market-orientation for the two advertising channels are discussed.

1 Introduction

Investment decision in advertising depends upon several factors, such as the typical characteristics of the customer whom the company addresses to, its business strategy and so on (Sutherland, 1993). We have tried to understand the different choices of investment in advertising of two market oriented telephone guide companies: the former (briefly G_A) with a long tradition and a very large experience in this field, the latter (G_B) which appeared recently. Nowadays we witness a different distribution of market-shares in advertising which is not steady yet. Analyzing the time series of a great number of small and medium companies that invested on advertising either in G_A or in G_B from 1997 to 2001, we have estimated a statistical model to identify the probability for a company to join the competitive group. The question may be presented like this: what probability has whatever company, which invested in G_A guide one year, to invest in G_B guide next year? The aim of the research is reached through the use of a simple stochastic model, based on Markov chains, to discover the various types of behaviour in the long period between the advertising policies and hence market-shares between the two competitors. Such a model was suggested by empirical experience based on the fact that, as the two competitors give the same service at the same price, the choice of the guide depends only on

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the recent experience of the companies (Dennison, 1994). The paper is divided into three parts: at first we illustrate the mathematical aspects of the model based on Markov theory to deal with the question mentioned above (Cox and Miller, 1965). Then, we get to an initial estimate the transition probabilities of the markovian process applied to the whole data set (Tweedie, 2001); finally, after stratifying the sample and estimating the transition matrix again, we make a prediction of what will happen to market-shares and business volumes of the two competitors.

2 The model

Letting X_t be the r.v. indicating the choice of a guide at year t, the sequence $\{X_t\}_{t=0,1,\ldots}$ can be described in terms of a simple discrete time homogeneous Markov process with a finite number of states S_j, with $j = 1, 2, 3$, defined as follows

S_1 the company invests in G_A;

S_2 the company invests in G_B;

S_3 the company does not invest.

This process is characterized by the property that the conditional distribution of X_t given $\{X_0, \ldots, X_{t-1}\}$ depends only on the value of X_{t-1} but no further on $\{X_0, \ldots, X_{t-2}\}$. In other words the process follows the Markov condition

$$P[X_t = S_j | X_0 = S_h, \ldots, X_{t-1} = S_i] = P[X_t = S_j | X_{t-1} = S_i]$$

The transitions between the states from two consecutive years are shown in Figure 1.

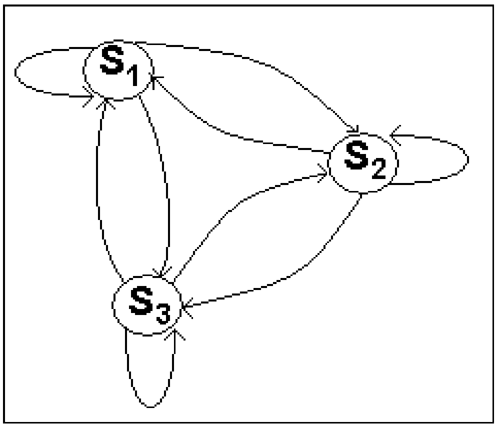

Figure 1: Passage from states.

The first order transition matrix for the Markov chain will take the simple form

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & 1 - p_{11} - p_{12} \\ p_{21} & p_{22} & 1 - p_{21} - p_{22} \\ p_{31} & p_{32} & 1 - p_{31} - p_{32} \end{bmatrix}$$

where $p_{ij} = P[X_t = S_j | X_{t-1} = S_i]$ and of course $\sum_{j=1}^{3} p_{ij} = 1$.

3 Some properties of the model

Let $\mathbf{p}^{(t)} = [p_1^{(t)}, p_2^{(t)}, p_3^{(t)}]$ be the row vector denoting the probabilities of finding the system in one of the three states at time t when the initial probabilities of the states are given by $\mathbf{p}^{(0)} = [p_1^{(0)}, p_2^{(0)}, p_3^{(0)}]$, then we have the following recurrence relations

$$p_1^{(t)} = p_1^{(t-1)}\, p_{11} + p_2^{(t-1)}\, p_{21} + p_3^{(t-1)}\, p_{31}$$

$$p_2^{(t)} = p_1^{(t-1)}\, p_{12} + p_2^{(t-1)}\, p_{22} + p_3^{(t-1)}\, p_{32}$$

$$p_3^{(t)} = p_1^{(t-1)}\, p_{13} + p_2^{(t-1)}\, p_{23} + p_3^{(t-1)}\, p_{33}$$

which may be written in the compact form

$$\mathbf{p}^{(t)} = \mathbf{p}^{(t-1)}\, \mathbf{P} \tag{3.1}$$

By iteration we get the main achievement of the model

$$\mathbf{p}^{(t)} = \mathbf{p}^{(t-1)}\, \mathbf{P} = \mathbf{p}^{(t-2)}\, \mathbf{P}^2 = \ldots = \mathbf{p}^{(0)}\, \mathbf{P}^{\mathbf{t}} \tag{3.2}$$

The previous relationship allows us to say that, given the initial probabilities $\mathbf{p}^{(0)}$ and the transition matrix $\mathbf{P}$, we may easily find the states occupation probabilities at any time t. Another interesting question is to establish whether after a sufficiently long period of years the system reaches a condition of statistical equilibrium. That is to say the state occupation probabilities would no longer depend on the initial conditions. In this case there is a stationary probability distribution $\boldsymbol{\pi}$=$[\pi_1, \pi_2, \pi_3]$ and, letting $t \uparrow \infty$ in (3.1), the vector $\boldsymbol{\pi}$ will satisfy

$$\boldsymbol{\pi} = \boldsymbol{\pi}\ \mathbf{P}$$

or equivalently, through the use of the identity matrix, you obtain

$$\boldsymbol{\pi}(\mathbf{I} - \mathbf{P}) = \mathbf{0}.$$

In order to find the stationary probability distribution of the process we have to solve the following system of four equations in three unknowns

$$\begin{cases} (1 - p_{11})\, \pi_1 - p_{21}\, \pi_2 - p_{31}\, \pi_3 & = & 0 \\ -p_{12}\, \pi_1 + (1 - p_{22})\, \pi_2 - p_{32}\, \pi_3 & = & 0 \\ -p_{13}\, \pi_1 - p_{23}\, \pi_2 + (1 - p_{33})\, \pi_3 & = & 0 \\ \pi_1 + \pi_2 + \pi_3 & = & 1 \end{cases} \tag{3.3}$$

The solutions of system (3.3), after applying Cramer method, are

$$\pi_1 = \frac{p_{21}\, p_{32} + p_{31} - p_{22}\, p_{31}}{D}$$

$$\pi_2 = \frac{p_{12}\, p_{31} + p_{32} - p_{11}\, p_{32}}{D}$$

$$\pi_3 = \frac{1 - p_{22} - p_{11} + p_{11}\,p_{22} - p_{12}\,p_{21}}{D}$$

where

$$D = det \begin{bmatrix} 1 & 1 & 1 \\ 1 - p_{11} & -p_{21} & -p_{31} \\ -p_{12} & 1 - p_{22} & -p_{32} \end{bmatrix}$$

4 A first estimate of the transition matrix

The question mentioned above has been investigated empirically through the application of Markov stochastic model to a huge collection of data: the time series of investments on advertisement of 32,219 companies on either G_A or in G_B from 1997 to 2001. For example, if we consider the investment choice (G_A, G_B, No Investment NI) operated by the companies for year 1998, known their situation in the previous year, we have the following distribution

$97 \rightarrow 98$	G_A	G_B	NI	$n_{i.}$
G_A	14,640	3,660	963	19,263
G_B	59	408	24	491
NI	997	2,867	8,601	12,465

Focusing on the choice on advertising in '98 for all companies that chose G_A in '97 we obtain the following conditional distribution

$97 \rightarrow 98$	G_A	G_B	NI
G_A	.76	.19	.05

Iterating this procedure for all subsequent years and for all other conditional distributions we have

	G_A	G_B	NI
G_A 97-98	.76	.19	.05
G_A 98-99	.77	.17	.06
G_A 99-00	.76	.17	.07
G_A 00-01	.74	.20	.06
mean	.7575	.1825	.0600

	G_A	G_B	NI
G_B 97-98	.12	.83	.05
G_B 98-99	.15	.82	.03
G_B 99-00	.13	.80	.07
G_B 00-01	.15	.79	.06
mean	.1375	.8100	.0525

	G_A	G_B	NI
NI 97-98	.08	.23	.69
NI 98-99	.10	.22	.68
NI 99-00	.08	.19	.73
NI 00-01	.09	.20	.71
mean	.0875	.2100	.7025

Conditional distributions seem to be stable through the years as shown in Figure 2.

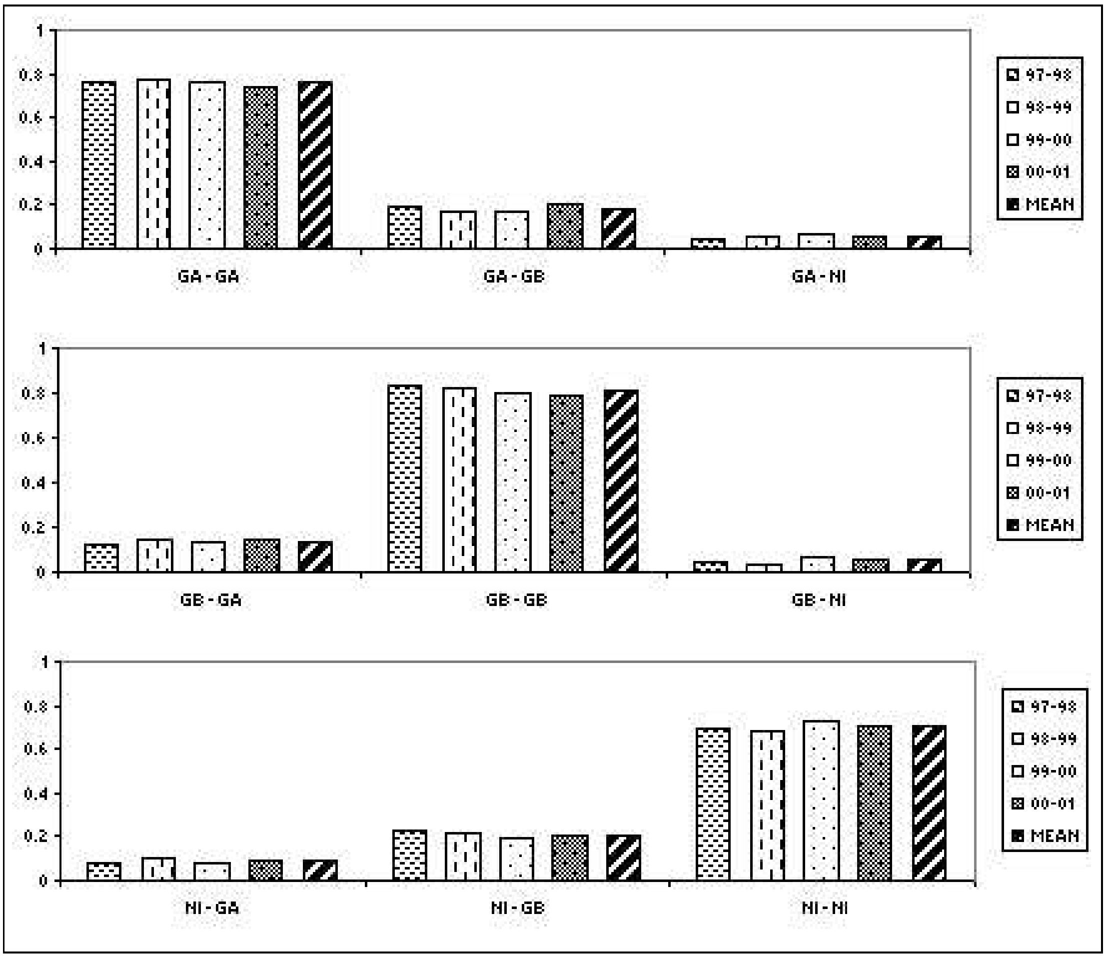

Figure 2: Transition probabilities.

Now we are going to figure how the competition between the two telephone guides will evolve in the next years. It's enough to remember that if you can estimate the vector of initial probabilities and the transition probability matrix you can also compute the probabilities of the states at any time t, referred to the coming years. Hence we have estimated the probabilities of the transition matrix using the mean of the conditional distributions over the four years considered previously.

$$\hat{\mathbf{P}} = \begin{bmatrix} \hat{p_{11}} & \hat{p_{12}} & \hat{p_{13}} \\ \hat{p_{21}} & \hat{p_{22}} & \hat{p_{23}} \\ \hat{p_{31}} & \hat{p_{32}} & \hat{p_{33}} \end{bmatrix} = \begin{bmatrix} .7575 & .1825 & .0600 \\ .1375 & .8100 & .0525 \\ .0875 & .2100 & .7025 \end{bmatrix} \tag{4.1}$$

While the initial probabilities of the states $\mathbf{p}^{(0)}$ have been estimated through the observed distribution of the companies starting from their choices of investment for the year 2001.

$$\hat{\mathbf{p}}^{(0)} = [0.47, 0.22, 0.31]$$

At this point we may evaluate which will be the state occupation probabilities in the future using equation (3.2) and the asymptotic distribution using the solution of the system (3.3)

$\mathbf{p}^{(t)} = \mathbf{p}^{(0)}\,\mathbf{P^t}$	year	G_A	G_B	NI
$\mathbf{p}^{(1)} = \mathbf{p}^{(0)}\,\mathbf{P^1}$	'02	0.41	0.33	0.26
$\mathbf{p}^{(2)} = \mathbf{p}^{(0)}\,\mathbf{P^2}$	'03	0.38	0.40	0.22
$\mathbf{p}^{(4)} = \mathbf{p}^{(0)}\,\mathbf{P^4}$	'05	0.36	0.44	0.20
$\boldsymbol{\pi} = \boldsymbol{\pi}\,\mathbf{P}$	stat.	0.34	0.50	0.16

So in the long period a company will tend to invest in G_A with probability 0.34 and in G_B with probability 0.50 and the most interesting aspect of this outcome is that it's independent by the initial state. Therefore we'll be able to say that the market-share of G_A will be around 34% (i.e. 10,954 companies) and the one of G_B will achieve 50% (i.e. 16,109 companies). Using these estimates and knowing that the average number of lines of advertising space bought by each company is 3.94, we may say that the business volume, measured in terms of lines, will be

G_A 43,160	G_B 63,471

These results are far from empirical expectation as G_A is older and more popular than the other telephone guide company. Since it's hard to believe that the business volume of G_A will be so reduced with respect to the one of G_B, and given that until now our analysis has been conducted on all the companies without taking into account any peculiar characteristic of them, such their dimension, we are going to explore this way in the following too.

5 Stratifying the sample

Exploiting the available information, we have stratified the sample in two groups: the small and the medium companies, according to the number of their employees.

Small (less than 30 employees)	20,626
Medium (more than 30 employees)	11,593

Then for the Small companies the transition probability matrix has been estimated as before

$$\hat{\mathbf{P}}_S = \begin{bmatrix} .5264 & .3721 & .1015 \\ .1485 & .8116 & .0399 \\ .0973 & .2347 & .6680 \end{bmatrix} \tag{5.1}$$

and similarly for the Medium companies

$$\hat{\mathbf{P}}_M = \begin{bmatrix} .9355 & .0518 & .0127 \\ .0715 & .9060 & .0225 \\ .0814 & .0311 & .8875 \end{bmatrix} \tag{5.2}$$

After solving the system (3.3) twice, separately for each group of data, we have obtained the following stationary distributions

$$\boldsymbol{\pi}_S = [0.23, 0.63, 0.14]$$

This suggests that only 23% of Small companies (i.e. 4,743 companies) will contribute to the market-share of G_A, while G_B will capture 63% of Small companies (i.e. 12,990 companies). If we focus on the medium companies an opposite situation appears

$$\boldsymbol{\pi}_M = [0.54, 0.34, 0.12]$$

The market-share of G_A will be around 54% of Medium companies (i.e. 6,260 companies) and the one of G_B will achieve 34% of Medium companies (i.e. 3,942 companies). These results suggest that the two telephone guide groups seem to be market-oriented in a different way: G_B addresses itself to Small companies, while G_A mainly to Medium companies. What can we finally say about the business volume of the two guides? Knowing that the average number of lines of advertising space bought by each company is 1.5 for Small companies and 8.3 for Medium companies, we can briefly summarize the business of G_A and G_B

	G_A	G_B
# lines Small	7,114	19,486
# lines Medium	51,960	32,715
# lines total	59,074	52,201

These results show that despite G_B will have a market-share greater than the one of G_A its business volume (in terms of number of lines) will result lower than the one of the competitor, because of being positioned on the market of Small companies.

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