Metodološki zvezki, Vol. 3, No. 1, 2006, 1-7

Ratio of Two Random Variables: A Note on the Existence of its Moments

Anton Cedilnik[1], Katarina Košmelj[2], and Andrej Blejec[3]

Abstract

To enable correct statistical inference, the knowledge about the existence of moments is crucial. The objective of this paper is to study the existence of the moments for the ratio $Z = X/Y$, where X and Y are arbitrary random variables with the additional assumption $P(Y = 0) = 0$. We present three existence theorems showing that specific behaviour of the distribution of Y in the neighbourhood of zero is essential. Simple consequences of these theorems give evidence to the existence of the moments for particular random variables; some of these results are well known from standard probability theory. However, we obtain them in a simple way.

1 Introduction

The ratio of two normally distributed random variables occurs frequently in statistical analysis. From standard probability literature, see for example Johnson *et al*. (1994), it is known that the ratio of two centred normal variables $Z = X/Y$ is a non-centred Cauchy variable. Marsaglia (1965) and Hinkley (1969) discussed the general situation: $[X \ \ Y]^{\mathrm{T}} \sim \mathrm{N}(\mu_X, \mu_Y, \sigma_X, \sigma_Y, \ \rho \neq \pm 1)$. Cedilnik *et al*. (2004) studied the general situation as well, they followed the same procedure as Hinkley did. They showed that the density of the ratio of two arbitrary normal variables can be expressed very neatly as a product of two factors.

[1] Biotechnical Faculty, University of Ljubljana, Jamnikarjeva 101, 1000 Ljubljana, Slovenia; anton.cedilnik@bf.uni-lj.si

[2] Biotechnical Faculty, University of Ljubljana Jamnikarjeva 101, 1000 Ljubljana, Slovenia; katarina.kosmelj@bf.uni-lj.si

[3] National Institute of Biology, University of Ljubljana, Večna pot 111, 1000 Ljubljana, Slovenia; andrej.blejec@nib.si

The first factor, the *Cauchy part*, is the density for a non-centred Cauchy variable, $\mathrm{C}\left(a=\rho\frac{\sigma_X}{\sigma_Y}, b=\frac{\sigma_X}{\sigma_Y}\sqrt{1-\rho^2}\right)$, which is independent of the expected values μ_X and μ_Y. The second factor, the *deviant part*, is a complicated function of *z* (see Cedilnik *et al*., 2003).

For illustration, let us consider the $\mathrm{N}(\mu_X, \mu_Y, \sigma_X = \sigma_Y = 1, \rho = 0.5)$. We vary μ_X from -4 to +4 with the step of 2, and μ_Y from 0 to 2 with the step of 0.5. The Cauchy part is the same for all these cases. The deviant part, however, takes different shapes. Figure 1a displays the Cauchy and the deviant part, Figure 1b the density which is their product.

It should be pointed out that the ratio of two normally distributed random variables has no moments due to the fact that the asymptotic behaviour of the density is the same as that of the Cauchy part.

To enable correct statistical inference the knowledge about the existence of the moments of the ratio is crucial. The objective of this paper is to study the existence of the moments of the ratio *for the general setting*.

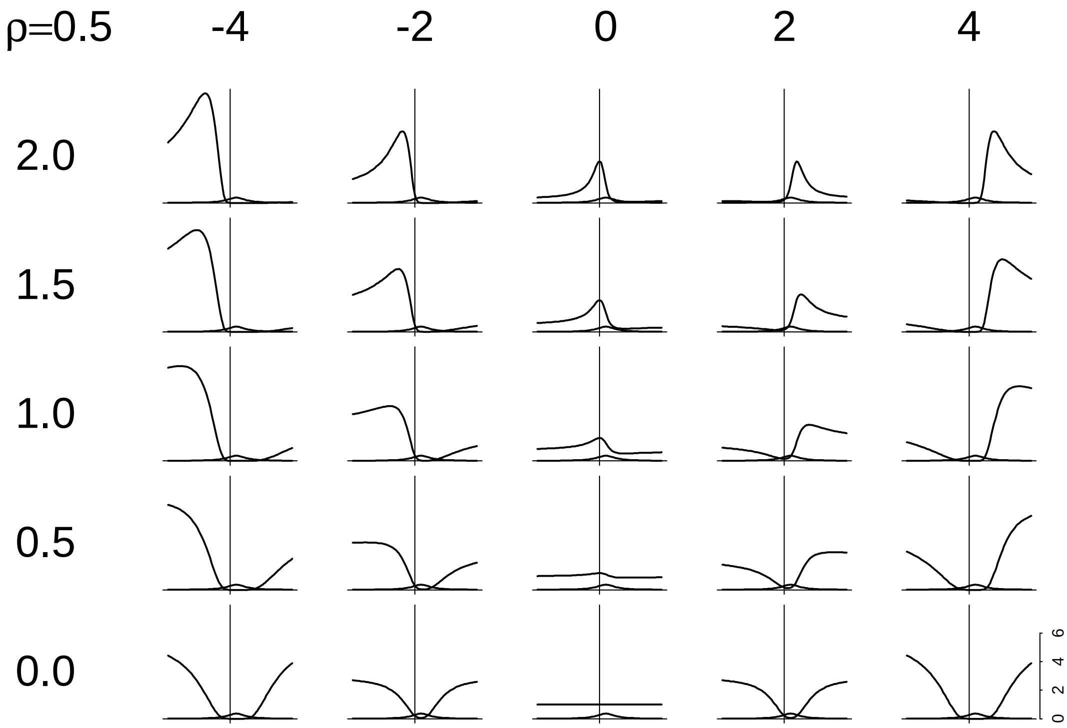

Figure 1a: The Cauchy part and the deviant part for the ratio X/Y, where $[X \;\; Y]^{\mathrm{T}} \sim \mathrm{N}(\mu_X, \mu_Y, \sigma_X = \sigma_Y = 1, \rho = 0.5)$. μ_X varies from -4 to +4 with the step of 2 (horizontally) and μ_Y from 0 to 2 with the step of 0.5 (vertically). The Cauchy part is constant for all these cases. For $\mu_X = \mu_Y = 0$ the deviant part equals 1.

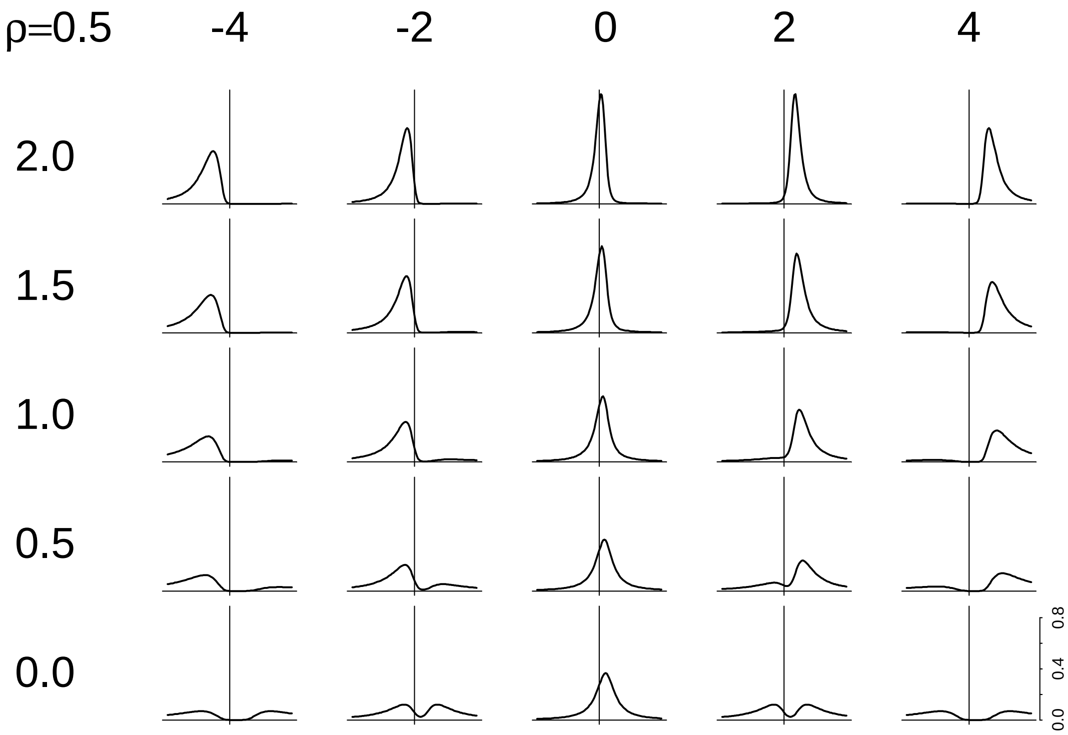

Figure 1b: The density for the ratio *X/Y*, where $[X \quad Y]^{\mathrm{T}} \sim$ $\mathrm{N}(\mu_X, \mu_Y, \sigma_X = \sigma_Y = 1, \rho = 0.5)$. μ_X varies from -4 to +4 with the step of 2 (horizontally) and μ_Y from 0 to 2 with the step of 0.5 (vertically).

2 Existence of the moments of the ratio

In what follows, we consider the random vector $[X \quad Y]^{\mathrm{T}}$, where X and Y are *arbitrary* random variables with the additional assumption $P(Y = 0) = 0$. Consequently, the ratio $Z = X/Y$ is a well defined random variable. We present three theorems on the existence of the moments and some consequences. The proofs are based on the well known Hölder's inequality which we present in Appendix to put on view the notations used in the text.

Theorem 1. *Suppose that there exists such an* $\varepsilon > 0$, *that* $P(|Y| < \varepsilon) = 0$. *If* X *has the moments of order* $\leq m$, m *nonnegative (possibly not integer), then* $Z = X/Y$ *has the moments of the same order. The following relationship holds*:

$$E\left(|Z|^m\right) \leq \varepsilon^{-m} E\left(|X|^m\right) \tag{2.1}$$

Proof. It is obvious. But just for the sake of uniform treatment, consider Hölder's inequality for $U = X^m$ and $V = Y^{-m}$. Since $P(-\varepsilon^{-1} \le Y^{-1} \le \varepsilon^{-1}) = 1$ and $P(-\varepsilon^{-m} \le Y^{-m} \le \varepsilon^{-m}) = 1$, $\operatorname{ess\,sup}\left|Y^{-m}\right| \le \varepsilon^{-m}$. **QED**

A simple consequence of this theorem is the fact that any ratio X/Y, where Y is discrete and does not have 0 as an adherent point, inherits the existence of the moments from *X*.

Theorem 2. *Let m, n be nonnegative (possibly not integer) numbers, $m > n$, and let X have the moments of order $\le m$. Further assume there exist two positive real numbers ε and C, such that for any δ within the interval $0 < \delta < \varepsilon$, the following holds:*

$$P\left(|Y| < \delta\right) \le C \cdot \delta^{\frac{mn}{m-n}\cdot(1+\varepsilon)} \tag{2.2}$$

Then $Z = X/Y$ has all the moments of order $\le n$.

Proof. For $n = 0$, the proof is trivial. For $n > 0$, use Hölder's inequality assuming $U = X^n$, $V = Y^{-n}$, and $p = \dfrac{m}{n}$, $q = \dfrac{m}{m-n}$:

$$\int_\Omega |Z|^n dP \le \left[\int_\Omega |X|^m dP\right]^{\frac{n}{m}} \cdot \left[\int_\Omega |Y|^{-\frac{mn}{m-n}} dP\right]^{\frac{m-n}{m}} .$$

Since X has the moments of order *m*, the first factor on the right is finite. Let us consider the second factor, denoting $T = |Y|^{-\frac{mn}{m-n}}$. Estimation gives (2.2):

$$P\left(T > \delta^{-\frac{mn}{m-n}}\right) = P\left(T^{-\frac{m-n}{mn}} < \delta\right) = P\left(|Y| < \delta\right) \le C \cdot \delta^{\frac{mn}{m-n}(1+\varepsilon)}$$

Hence, $P(T > t) \le C \cdot t^{-1-\varepsilon}$ for any *t* which is large enough. We show further that under (2.2) the second factor is finite

$$\int_\Omega |Y|^{-\frac{mn}{m-n}} dP = \int_{0 \le T \le 2^N} T dP + \sum_{j=0}^{\infty} \int_{2^{N+j} < T \le 2^{N+j+1}} T dP$$

$$\le 2^N \cdot P(0 \le T \le 2^N) + \sum_{j=0}^{\infty} 2^{N+j+1} \cdot P(2^{N+j} < T \le 2^{N+j+1})$$

$$\le 2^N + \sum_{j=0}^{\infty} 2^{N+j+1} \cdot P(T > 2^{N+j})$$

$$\le 2^N + \sum_{j=0}^{\infty} 2^{N+j+1} \cdot C \cdot 2^{-(N+j)(1+\varepsilon)} < \infty,$$

for N large enough. **QED**

The expression (2) is very general because it holds for an arbitrary random variable *Y*. Example 1 presents its form for continuous random variable *Y*, Example 2 provides the discrete case.

Example 1. Assume that X has the moments of order $\le m$, and Y is continuously distributed with the density

$v(y) \le A \cdot |y|^a$ ($A > 0$, $a > -1$) for $-\varepsilon < y < \varepsilon$, else it is arbitrary.

Then:

$$P(|Y| < \delta) = \int_{-\delta}^{\delta} v(y)dy \le 2\int_0^{\delta} Ay^a dy = \frac{2A}{1+a}\delta^{1+a} .$$

From (2.2) we get

$$n = \frac{m}{1 + \frac{1+\varepsilon}{1+a} m} .$$

Therefore, Z has all the moments of order less than $\dfrac{m}{1+\frac{m}{1+a}}$.

In the light of this example some results on the existence of the moments for particular random variables appear which are well known from standard probability theory. For illustration we present two of them.

Example 1a. As a simple consequence of Example 1 consider Y which is normally distributed. Then $a = 0$; consequently, $\dfrac{m}{1+\frac{m}{1+a}} < 1$; therefore no moments of Z exist. A special case is mentioned in Introduction: the ratio of two normally distributed random variables has no moments.

Example 1b. Consider $X \sim \mathrm{N}(0, 1)$, $U \sim \chi^2(r)$ and the ratio $T = \dfrac{X}{\sqrt{U}}\sqrt{r}$**,** a Student variable with *r* degrees of freedom. It is well known that $Y = \sqrt{U}$ has a chi-distribution with *r* degrees of freedom with the probability density

$$v(y) = \frac{1}{2^{r/2-1}\Gamma\left(\frac{r}{2}\right)} y^{r-1} \exp\left(-\frac{y^2}{2}\right) \quad \text{(for } y \ge 0 \text{)}$$

which can be estimated in light of Example 1:

$$v(y) \;\le\; \frac{1}{2^{\,r/2-1}\Gamma\!\left(\frac{r}{2}\right)}|y|^{r-1} \qquad \text{(for } \textit{any} \text{ real } y\text{)}.$$

Then T has all moments of order less than $\dfrac{m}{1+\frac{m}{r}}$ (where m is arbitrary), which is as close to r as one wants. Hence, T has all the moments of order less than r, as it is well known from standard probability theory.

Example 2. Let Y have a discrete distribution with a strictly decreasing infinite sequence $(a_j)\downarrow 0$ of values, and $P(Y=a_j)=v_j$. Assume that there exist two positive real numbers ε and C such that if $a_N<\varepsilon$ the following holds:

$$\sum_{j=N+1}^{\infty} v_j \;\le\; C\cdot (a_N)^{\frac{mn}{m-n}(1+\varepsilon)} \tag{2.3}$$

Than (2.2) is valid.

The next theorem describes the reverse situation: we study the existence of the moments of *X* given the moments for *Z* and *Y*.

Theorem 3. *If* Y *has the moments of order* m *and* Z *the moments of order* n, m *and* n *are positive (possibly not integers), then* X *has the moments of order* $j\le \dfrac{mn}{m+n}$. *The following relationship holds:*

$$E\left(|X|^{j}\right)\le E\left(|Y|^{jp}\right)^{1/p}\cdot E\left(|Z|^{jq}\right)^{1/q}, \tag{2.4}$$

where $p>1$, $1/p+1/q=1$ *and* $0\le j\le \min\left\{\frac{m}{p},\ \frac{n}{q}\right\}$.

Proof. The case $j=0$ is trivial. Let us use Hölder's inequality for $U=Y^j$ and $V=Z^j$ for some $j>0$. The estimation has sense if $jp\le m$ and $jq\le n$. Hence, $j\le \min\left\{\frac{m}{p},\frac{n}{q}\right\}\le \max_p \min\left\{\frac{m}{p},\frac{n}{q}\right\} = \dfrac{mn}{m+n}$; the maximum is reached at $p=\dfrac{m+n}{n}$.

Appendix

Hölder's inequality: Let (Ω, F, P) be a probability space, where Ω is a set of outcomes g, F a Borel's σ-algebra of events, and P the probability measure. Further, let U and V be random variables on Ω. Then

$$\int_\Omega |U(g)V(g)| dP \quad \leq \quad \left[\int_\Omega |U(g)|^p \, dP \right]^{1/p} \cdot \left[\int_\Omega |V(g)|^q \, dP \right]^{1/q}$$

for any pair of positive p, q, where $1/p + 1/q = 1$, if the integrals on the right converge. If $p = 1$ and $q = \infty$, then

$$\int_\Omega |U(g)V(g)| dP \quad \leq \quad \int_\Omega |U(g)| dP \cdot \text{ess sup}_{g \in \Omega} |V(g)|$$

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