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Some New Construction Methods of Variance Balanced Block Designs with Repeated Blocks

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Abstract

Some new construction methods of the variance balanced block designs with repeated blocks are given. They are based on the specialized product of incidence matrices of the balanced incomplete block designs.

1 Introduction

In the paper we present some types of block designs, which are used in practice as well as in the general theory of block designs. For a variety of reasons, it is desirable to have the balanced incomplete block design with the block repetitions, because it might be less expensive and easier to implement. In many applications, the experimenter may not wish to run certain treatment combinations. For example, it is physically impossible to run three or more treatments combinations in one block. However, this combination may produce observations which no longer conform to the homoscedastic linear model. Foody and Hedayat (1977) present some potential applications of the balanced incomplete block designs with repeated blocks to experimental designs and controlled sampling. Designs with repeated blocks with the equireplications and with equal size of each block are discussed in the literature: Hedayat and Li (1979), Hedayat and Hwang (1984). However from a practical point of view, it may not be possible to construct a design with equiblock sizes accommodating the equireplication of each treatment in all the blocks. Here we consider a class of block designs called variance balanced block designs which can be made available in unequal block sizes and for equal replications. In the variance balanced block design each elementary contrast is estimated with the same variance.

Let us consider v treatments arranged in b blocks in a block design with incidence matrix $\mathbf{N} = [n_{ij}],\ i = 1, 2, ..., v,\ j = 1, 2, ..., b$, where n_{ij} denotes the number of experimental units in the jth block getting the ith treatment, $n = \sum_{i=1}^{v} \sum_{j=1}^{b} n_{ij}$. When $n_{ij} = 1$ or 0 for all i and j, the design is said to be binary. Otherwise, it is said to be nonbinary. In this paper we consider binary block designs, only. The following notation is used: $\mathbf{r} = [r_1, r_2, ..., r_v]'$ is the vector of treatment replications, $\mathbf{k} = [k_1, k_2, ..., k_b]'$ is the vector of block sizes. Hence, $\mathbf{N}\mathbf{1}_b = \mathbf{r}$ and $\mathbf{N}'\mathbf{1}_v = \mathbf{k}$, where $\mathbf{1}_a$ is the $a \times 1$ vector of

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ones.
The information matrix $\mathbf{C}$ for treatment effects is defined as

$$\mathbf{C} = \mathbf{R} - \mathbf{N}\mathbf{K}^{-1}\mathbf{N}', \tag{1.1}$$

where $\mathbf{R} = \mathrm{diag}\,[r_1, r_2, .., r_v]$, $\mathbf{K} = \mathrm{diag}\,[k_1, k_2, ..., k_b]$. The information matrix $\mathbf{C}$ is very suitable in determining properties of a block designs.
For several reasons, in particular from a practical point of view, it is desirable to have repeated blocks in the design. For example, some treatment combinations may be preferable over others, and also the design implementation may cost differently according to the design structure contains or not repeated blocks. The set of all distinct blocks in a block design is called the support of the design and the cardinality of the support is denoted by b^* and is referred to as the support size of the design.
In the literature, see Caliński (1977), Puri and Nigam (1977), there are considered the balanced designs in various senses. In present paper we consider a balanced design of the following type, given in Rao (1958). A block design is said to be balanced if every elementary contrast of treatment effects is estimated with the same variance. In this sense the design is also called a variance balanced (VB) block design.
It is well known that a block design is a VB if and only if it has

$$\mathbf{C} = \eta \left[\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}_v' \right], \tag{1.2}$$

where η is the unique nonzero eigenvalue of the $\mathbf{C}-$ matrix with multiplicity $v-1$, $\mathbf{I}_v$ is the $v \times v$ identity matrix. For a binary block design

$$\eta = \frac{\sum_{i=1}^{v} r_i - b}{v-1}$$

(see Kageyama and Tsuji (1979)).
In the particular case, when the block design is equireplicated, then $\eta = \frac{vr-b}{v-1}$.

2 Construction of the design matrices

Now, we consider balanced incomplete block design (BIBD) (See Raghavarao (1971)) as an arrangement of v treatments into b blocks each of k $(< v)$ treatments, satisfying conditions: every treatment occurs at most once in each block and occurs in r blocks, every pair of treatments occurs together in λ blocks. The parameters of the BIBD are $v,\ b,\ r,\ k,\ \lambda$ and they satisfy

$$vr = bk, \qquad \lambda(v-1) = r(k-1).$$

Let $\mathbf{N}$ be an incidence matrix of the BIBD. We have, $\mathbf{N}\mathbf{N}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$. It is known from literature, that any BIBD is VB.

Theorem 1 *If* $\mathbf{N}_i$ *is BIBD with parameters* $v,\ b_i,\ r_i,\ k_i,\ \lambda_i$ *and* $\mathbf{C}_i$ *is* $\mathbf{C}-$ *matrix for* $i = 1, 2, ..., t,$ *then*

$$\mathbf{N} = [\mathbf{N}_1\ \ \mathbf{N}_2\ \ ...\ \ \mathbf{N}_t] \tag{2.1}$$

is the incidence matrix of the VB block design.

Proof. For the design $\mathbf{N}$ in (2.1), we have $r = \sum_{i=1}^{t} r_i,$ $\mathbf{k} = \left[k_1\mathbf{1}'_{b_1}\ \ k_2\mathbf{1}'_{b_2}\ \ ...\ \ k_t\mathbf{1}_{b_t}\right]'$. Thus

$\mathbf{C} = r\mathbf{I}_v - \mathbf{N}\mathbf{K}^{-1}\mathbf{N}' = r\mathbf{I}_v - \sum_{i=1}^{t} \frac{1}{k_i}\mathbf{N}_i\mathbf{N}'_i = \sum_{i=1}^{t} r_i\mathbf{I}_v - \sum_{i=1}^{t} \frac{1}{k_i}\mathbf{N}_i\mathbf{N}'_i$

$= \sum_{i=1}^{t} \left(r_i\mathbf{I}_v - \frac{1}{k_i}\mathbf{N}_i\mathbf{N}'_i\right) = \sum_{i=1}^{t} \mathbf{C}_i.$

The design $\mathbf{N}_i$ is VB as BIBD. Therefore from (1.2), we have

$\mathbf{C} = \sum_{i=1}^{t} \eta_i \left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}'_v\right) = \eta \left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}'_v\right)$, where $\eta = \sum_{i=1}^{t} \eta_i.$

Hence, the claim of the Theorem.

We use the following specialized product of two matrices presented in Pal and Dutta (1979). If $\mathbf{A} = (a_{st})_{m\times p}$ and $\mathbf{B} = (b_{zl})_{m\times q},$ then the specialized product of the matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{D} = \mathbf{A} * \mathbf{B} = (d_{sl})_{m\times pq}, \tag{2.2}$$

where $d_{sl} = a_{st} \times b_{iz}$, l being equal to $(t-1)q + z$ for $s = 1, 2, ..., m,\ t = 1, 2, ..., p,\ z = 1, 2, ..., q.$

Let $\mathbf{N}_i,\ i = 1, 2,$ be an incidence matrix of the BIBD with parameters $v,\ b_i,\ r_i,$ $k_i,\ \lambda_i$. Let $\mathbf{C}_i$ be the $\mathbf{C}-$matrix of this design defined by $\mathbf{N}_i$. Now, we form the matrix $\mathbf{N}$ as

$$\mathbf{N} = \mathbf{N}_1\ *\ \mathbf{N}_2. \tag{2.3}$$

Theorem 2 *If* $\mathbf{N}_1$ *is an incidence matrix of the BIBD with parameters* $v,\ \ b_1 = v(v-1)/2,\ \ r_1 = v-1,\ \ k_1 = 2,\ \ \lambda_1 = 1$ *and* $\mathbf{N}_2$ *is an incidence matrix of the BIBD with parameters* $v = b_2,\ \ r_2 = k_2 = v-1,\ \ \lambda_2 = v-2,$ *then* $\mathbf{N}$ *in the form (2.3) is an incidence matrix of the VB block design with repeated blocks and with parameters*

$$v,\quad b = v^2(v-1)/2,\quad r = (v-1)^2,\quad \mathbf{k} = \begin{bmatrix} 2\cdot\mathbf{1}_{v(v-1)(v-2)/2} \\ \mathbf{1}_{v(v-1)} \end{bmatrix},\quad b^* = v(v+1)/2.$$

Proof. For the product (2.3) to hold, we have $\mathbf{N} = \left[\mathbf{N}_1 \otimes \mathbf{1}'_{v-2}\quad \mathbf{I}_v \otimes \mathbf{1}'_{v-1}\right]$. Hence, the information matrix $\mathbf{C} = (v-2)\mathbf{C}_1.$ Therefore, taking into consideration Theorem 1, $\mathbf{N}$ is an incidence matrix of the VB block design with repeated blocks. So, the Theorem is proven.

Let us consider the class of BIBD's for $k = 3$ and $\lambda = 1$ usually known as Steiner's triple system. There are only two series of Steiner's triple systems (See Raghavarao (1971)) with respective parameters for $t = 1, 2, ...$

$$v = 6t + 1, \quad b = t(6t + 1), \quad r = 3t, \quad k = 3, \quad \lambda = 1 \tag{2.4}$$

$$v = 3(2t + 1), \quad b = (2t + 1)(3t + 1), \quad r = 3t + 1, \quad k = 3, \quad \lambda = 1. \tag{2.5}$$

Steiner (1853) posed the problem whether the two series of BIBD's with parameters given in (2.4) and (2.5) exist for every t and later on Moore (1893) and Hanani (1961) recursive methods of constructing Steiner's triple systems for all t are given. For a detailed account of showing the existence of such designs see Hall, Jr. (1967).

Theorem 3 *If* $\mathbf{N}_1$ *is an incidence matrix of the BIBD with parameters given in (2.4) and* $\mathbf{N}_2$ *is an incidence matrix of the BIBD with parameters* $v = b_2 = 6t + 1, \quad r_2 = k_2 = 6t, \quad \lambda_2 = 6t - 1,$ *then* $\mathbf{N}$ *in the form (2.3) is an incidence matrix of the VB block design with repeated blocks and with parameters*
$v, \quad b = t(6t + 1)^2, \quad r = 18t^2, \quad \mathbf{k} = \begin{bmatrix} 3 \cdot \mathbf{1}_{2t(3t-1)(6t+1)} \\ 2 \cdot \mathbf{1}_{3t(6t+1)} \end{bmatrix}, \quad b^* = 4t(6t + 1).$

Proof. For the product (2.3), we have $\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \otimes \mathbf{1}'_{2(3t-1)} & \mathbf{N}_3 \end{bmatrix}$, where $\mathbf{N}_3$ is an incidence matrix of BIBD with parameters $v = 6t + 1, \ b_3 = 3t(6t + 1), \ r_3 = 6t, \ k_3 = 2, \ \lambda_3 = 1$ and with information matrix $\mathbf{C}_3$. Thus $\mathbf{C} = 2(3t - 1)\mathbf{C}_1 + \mathbf{C}_3$. Therefore, taking into consideration Theorem 1, $\mathbf{N}$ is an incidence matrix of the VB block design with repeated blocks. Hence, the result.

Theorem 4 *If* $\mathbf{N}_1$ *is an incidence matrix of the BIBD with parameters given in (2.5) and* $\mathbf{N}_2$ *is an incidence matrix of the BIBD with parameters* $v = b_2 = 3(2t + 1), \quad r_2 = k_2 = 2(3t + 1), \quad \lambda_2 = 6t + 1,$ *then* $\mathbf{N}$ *in the form (2.3) is an incidence matrix of the VB block design with repeated blocks and with parameters*
$v, \quad b = 3(2t + 1)^2(3t + 1), \quad r = 2(3t + 1)^2, \quad \mathbf{k} = \begin{bmatrix} 3 \cdot \mathbf{1}_{6t(2t+1)(3t+1)} \\ 2 \cdot \mathbf{1}_{3(2t+1)(3t+1)} \end{bmatrix},$
$b^* = 4(2t + 1)(3t + 1).$

Proof. For the product (2.3), we have $\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \otimes \mathbf{1}'_{6t} & \mathbf{N}_4 \end{bmatrix}$, where $\mathbf{N}_4$ is an incidence matrix of BIBD with parameters $v = 3(2t + 1), \ b_4 = 3(2t + 1)(3t + 1), \ r_4 = 2(3t + 1), \ k_4 = 2, \ \lambda_4 = 1$ and with the information matrix $\mathbf{C}_4$. Hence the information matrix $\mathbf{C}$ is given as $\mathbf{C} = 6t\mathbf{C}_1 + \mathbf{C}_4$. Owing to Theorem 1 it implies, that $\mathbf{N}$ is an incidence matrix of the VB block design with repeated blocks. So, the Theorem is proven.

Theorem 5 *If* $\mathbf{N}_1$ *is an incidence matrix of the symmetrical BIBD with parameters* $v = b_1, \ r_1 = k_1, \ \lambda_1,$ *then* $\mathbf{N}$ *in the form*

$$\mathbf{N} = \mathbf{N}_1 \ * \ \mathbf{N}_1 \tag{2.6}$$

is an incidence matrix of the VB block design with repeated blocks and with parameters $v,\ b = v^2,\ r = r_1^2,\ \mathbf{k} = \begin{bmatrix} k_1\mathbf{1}_v \\ \lambda_1\mathbf{1}_{v(v-1)} \end{bmatrix},\ b^* = v(v+1)/2.$

Proof. For the product (2.6), we have $\mathbf{N} = [\mathbf{N}_1\ \mathbf{N}_5\ \mathbf{N}_5]$, where $\mathbf{N}_5$ is an incidence matrix of BIBD with parameters v, $b_5 = v(v-1)/2$, $r_5 = r_1(r_1-1)/2$, $k_5 = \lambda_1$, $\lambda_5 = (r_1(r_1-1)(\lambda_1-1))\,/\,(2(v-1))$ and with information matrix $\mathbf{C}_5$. Thus the information matrix $\mathbf{C} = \mathbf{C}_1 + 2\mathbf{C}_5$. That means $\mathbf{N}$ is an incidence matrix of the VB block design with repeated blocks, because of Theorem 1. Hence, the claim of the Theorem holds.

Corollary 1 *If* $\mathbf{N}_1$ *is an incidence matrix of the symmetrical BIBD with parameters* $v = b_1 = 4t+3$, $r_1 = k_1 = 2(t+1)$, $\lambda_1 = t+1$, $4t+3$ *is a prime or a prime power, then* $\mathbf{N}$ *in the form (2.6) is an incidence matrix of the VB block design with repeated blocks and with parameters* v, $b = (4t+3)^2$, $r = 4(t+1)^2$,
$\mathbf{k} = \begin{bmatrix} 2(t+1)\cdot\mathbf{1}_{4t+3} \\ (t+1)\cdot\mathbf{1}_{2(2t+1)(4t+3)} \end{bmatrix}$, $b^* = 2(t+1)(4t+3)$.

Corollary 2 *If* $\mathbf{N}_1$ *is an incidence matrix of the symmetrical BIBD with parameters* $v = b_1 = 4t^2$, $r_1 = k_1 = t(2t+1)$, $\lambda_1 = t(t+1)$, *then* $\mathbf{N}$ *in the form (2.6) is an incidence matrix of the VB block design with repeated blocks and with parameters* v, $b = 16t^4$, $r = t^2(2t+1)^2$, $\mathbf{k} = \begin{bmatrix} t(2t+1)\cdot\mathbf{1}_{4t^2} \\ t(t+1)\cdot\mathbf{1}_{4t^2(4t^2-1)} \end{bmatrix}$, $b^* = 2t^2(4t^2+1)$.

Corollary 3 *If* $\mathbf{N}_1$ *is an incidence matrix of the symmetrical BIBD with parameters* $v = b_1 = 4t^2-1$, $r_1 = k_1 = 2t^2$, $\lambda_1 = t^2$, *then* $\mathbf{N}$ *in the form (2.6) is an incidence matrix of the VB block design with repeated blocks and with parameters* v, $b = (4t^2-1)^2$, $r = 4t^4$, $\mathbf{k} = \begin{bmatrix} 2t^2\cdot\mathbf{1}_{4t^2-1} \\ t^2\cdot\mathbf{1}_{2(2t^2-1)(4t^2-1)} \end{bmatrix}$, $b^* = 2t^2(4t^2-1)$.

Corollary 4 *If* $\mathbf{N}_1$ *is an incidence matrix of the symmetrical BIBD with parameters* $v = b_1 = 8t+7$, $r_1 = k_1 = 4(t+1)$, $\lambda_1 = 2(t+1)$, *then* $\mathbf{N}$ *in the form (2.6) is an incidence matrix of the VB block design with repeated blocks and with parameters* v, $b = (8t+7)^2$, $r = 16(t+1)^2$, $\mathbf{k} = \begin{bmatrix} 4(t+1)\cdot\mathbf{1}_{8t+7} \\ 2(t+1)\cdot\mathbf{1}_{2(4t+3)(8t+7)} \end{bmatrix}$,
$b^* = 4(t+1)(8t+7)$.

Corollary 5 *If* $\mathbf{N}_1$ *is an incidence matrix of the symmetrical BIBD with parameters* $v = b_1 = t^2+t+1$, $r_1 = k_1 = t^2$, $\lambda_1 = t(t-1)$, *where* t *is a prime or a prime power, then* $\mathbf{N}$ *in the form (2.6) is an incidence matrix of the VB block design with repeated blocks and with parameters* v, $b = (t^2+t+1)^2$, $r = t^4$,
$\mathbf{k} = \begin{bmatrix} t^2\cdot\mathbf{1}_{t^2+t+1} \\ t(t-1)\cdot\mathbf{1}_{t(t+1)(t^2+t+1)} \end{bmatrix}$, $b^* = (t^2+t+1)(t^2+t+2)/2$.

Corollary 6 *If* $\mathbf{N}_1$ *is an incidence matrix of the symmetrical BIBD with parameters* $v = b_1 = (t+1)(t^2+1)$, $r_1 = k_1 = t^3$, $\lambda_1 = t^2(t-1)$, *where* t *is a prime or a prime power, then* $\mathbf{N}$ *in the form (2.6) is an incidence matrix of the VB block design with repeated blocks and with parameters* v, $b = (t+1)^2(t^2+1)^2$, $r = t^6$, $\mathbf{k} = \begin{bmatrix} t^3 \cdot \mathbf{1}_{(t+1)(t^2+1)} \\ t^2(t-1) \cdot \mathbf{1}_{t(t+1)(t^2+1)(t^2+t+1)} \end{bmatrix}$, $b^* = (t+1)(t^2+1)(t^3+t^2+t+2)/2$.

Corollary 7 *If* $\mathbf{N}_1$ *is an incidence matrix of the symmetrical BIBD with parameters* $v = b_1$, $r_1 = k_1 = v-1$, $\lambda_1 = v-2$, *then* $\mathbf{N}$ *in the form (2.6) is an incidence matrix of the VB block design with repeated blocks and with parameters* v, $b = v^2$, $r = (v-1)^2$, $\mathbf{k} = \begin{bmatrix} (v-1) \cdot \mathbf{1}_v \\ (v-2) \cdot \mathbf{1}_{v(v-1)} \end{bmatrix}$, $b^* = v(v+1)/2$.

3 Conclusions and examples

The importance of block repetition in a design is very well known, so many authors pay special attention to the construction rules and a practical properties of designs having repeated blocks. So, we present appropriate examples of constructions of the design matrices.

Example 1 Let us consider the BIBD (See Theorem 2) with parameters $v = 4$, $b_1 = 6$, $r_1 = 3$, $k_1 = 2$, $\lambda_1 = 1$ with incidence matrix $\mathbf{N}_1$ and the BIBD with parameters $v = 4$, $b_2 = 4$, $r_2 = 3$, $k_2 = 3$, $\lambda_2 = 2$ with incidence matrix $\mathbf{N}_2$, where

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Based on the matrices $\mathbf{N}_1$ and $\mathbf{N}_2$, we form the incidence matrix $\mathbf{N}$ in the form (2.3) of the VB block design with repeated blocks and with parameters
$v = 4$, $b = 24$, $r = 9$, $\mathbf{k} = \begin{bmatrix} 2 \cdot \mathbf{1}_{12} \\ \mathbf{1}_{12} \end{bmatrix}$, $b^* = 10$,

$$\mathbf{N} = \begin{bmatrix} 1&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1&1&1&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&1&0&1 \\ 0&0&0&0&1&0&1&1&1&0&1&1&0&0&0&0&0&0&0&0&1&0&1&1 \\ 0&0&0&0&0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&1&0&0&0&0 \end{bmatrix}.$$

Hence, after permutation of columns, we have $\mathbf{N} = [\mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{I}_4 \quad \mathbf{I}_4 \quad \mathbf{I}_4]$.

Example 2 Let us consider the BIBD (See Theorem 3) with parameters $v = b_1 = 7$, $r_1 = k_1 = 3$, $\lambda_1 = 1$ with incidence matrix $\mathbf{N}_1$ and the BIBD with parameters $v = b_2 = 7$, $r_2 = k_2 = 6$, $\lambda_2 = 5$ with incidence matrix $\mathbf{N}_2$, where

$$\mathbf{N}_1 = \begin{bmatrix} 1&1&1&0&0&0&0\\ 1&0&0&1&1&0&0\\ 0&1&0&1&0&1&0\\ 1&0&0&0&0&1&1\\ 0&0&1&1&0&0&1\\ 0&0&1&0&1&1&0\\ 0&1&0&0&1&0&1 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} 0&1&1&1&1&1&1\\ 1&0&1&1&1&1&1\\ 1&1&0&1&1&1&1\\ 1&1&1&0&1&1&1\\ 1&1&1&1&0&1&1\\ 1&1&1&1&1&0&1\\ 1&1&1&1&1&1&0 \end{bmatrix}.$$

Based on the matrices $\mathbf{N}_1$ and $\mathbf{N}_2$, we form the incidence matrix $\mathbf{N}$ in the form (2.3) of the VB block design with repeated blocks and with parameters $v = 7,\ b = 49,\ r = 18,\ \mathbf{k} = \begin{bmatrix} 3\cdot\mathbf{1}_{28} \\ 2\cdot\mathbf{1}_{21} \end{bmatrix},\ b^* = 28$ and, after permutation of columns, $\mathbf{N} = [\mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{N}_3]$, where $\mathbf{N}_3$ is an incidence matrix of BIBD with parameters $v = 7,\ b_3 = 21,\ r_3 = 6,\ k_3 = 2,\ \lambda_3 = 1$,

$$\mathbf{N}_3 = \begin{bmatrix} 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0\\ 0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0\\ 0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1\\ 0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1 \end{bmatrix}.$$

Example 3 Let us consider the BIBD (See Theorem 4) with parameters $v = 9$, $b_1 = 12,\ r_1 = 4,\ k_1 = 3,\ \lambda_1 = 1$ with incidence matrix $\mathbf{N}_1$ and the BIBD with parameters $v = b_2 = 9,\ r_2 = k_2 = 8,\ \lambda_2 = 7$ with incidence matrix $\mathbf{N}_2$, where

$$\mathbf{N}_1 = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&0&0&0&1&1&1&0&0&0&0&0\\ 0&1&0&0&1&0&0&1&1&0&0&0\\ 0&0&1&0&0&1&0&1&0&1&0&0\\ 0&0&0&1&0&0&1&1&0&0&1&0\\ 1&0&0&0&0&0&0&0&1&1&1&0\\ 0&1&0&0&0&0&1&0&0&1&0&1\\ 0&0&1&0&1&0&0&0&0&0&1&1\\ 0&0&0&1&0&1&0&0&1&0&0&1 \end{bmatrix}, \mathbf{N}_2 = \begin{bmatrix} 0&1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&1\\ 1&1&1&0&1&1&1&1&1\\ 1&1&1&1&0&1&1&1&1\\ 1&1&1&1&1&0&1&1&1\\ 1&1&1&1&1&1&0&1&1\\ 1&1&1&1&1&1&1&0&1\\ 1&1&1&1&1&1&1&1&0 \end{bmatrix}.$$

Based on the matrices $\mathbf{N}_1$ and $\mathbf{N}_2$, we form the incidence matrix $\mathbf{N}$ in the form (2.3) of the VB block design with repeated blocks and with parameters $v = 9,\ b = 108,\ r = 32,\ \mathbf{k} = \begin{bmatrix} 3\cdot\mathbf{1}_{72} \\ 2\cdot\mathbf{1}_{36} \end{bmatrix},\ b^* = 48$ and, after permutation of columns, $\mathbf{N} = [\mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{N}_1 \quad \mathbf{N}_4]$, where $\mathbf{N}_4$ is an incidence matrix of BIBD with parameters $v = 9,\ b_4 = 36,\ r_4 = 8,\ k_4 = 2,\ \lambda_4 = 1$,

$$\mathbf{N}_4 = \left[\begin{array}{cccccccccccccccccccccccccccccccccccc}
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1
\end{array}\right].$$

References

[1] Caliński, T. (1977): On the notion of balance block designs. In G. Barra et al. (Eds.): *Recent Developments in Statistics*, 365-374. Amsterdam: North-Holland Publishing Company.

[2] Foody, W. and Hedayat, A. (1977): On theory and applications of BIB designs with repeated blocks. *Ann. Statist.,* **5**, 932-945.

[3] Hall, M., Jr. (1967): Combinatorial theory. Blaisdell Publishing Company.

[4] Hanani, H. (1961): The existence and construction of balanced incomplete block designs. *Ann. Math.,* **32**, 361-386.

[5] Hedayat, A. and Li Shuo-Yen, R. (1979): The trade off method in the construction of BIB designs with repeated blocks. *Ann. Statist.,* **7**, 1277-1287.

[6] Hedayat, A. and Hwang, H.L. (1984): BIB(8,56,21,3,6) and BIB(10,30,9,3,2) designs with repeated blocks. *J. Comb. Th.,* **A36**, 73-91.

[7] Kageyama, S. and Tsuji, T. (1979): Inequality for equireplicated n-array block designs with unequal block sizes. *Journal of Statistical Planning and Inferences,* **3**, 101-107.

[8] Moore, E.H. (1893): Concerning triple systems. *Math. Ann.,* **43**, 271-285.

[9] Pal, S. and Dutta, B.K. (1979): Some new methods of construction of efficiency balanced designs. *Calcutta Statist. Ass. Bull.,* **28**, 157-161.

[10] Puri, P.D. and Nigam, A.K. (1977): Balanced block designs. *Common. Statist.-Theor. Meth.,* **A6**, 1171-1179.

[11] Raghavarao, D. (1971): Constructions and combinatorial problems in design of experiments. New York: John Wiley and Sons, Inc.

[12] Rao, V.R. (1958): A note on balanced designs. *Ann. Math. Statist.,* **29**, 290-294.

[13] Steiner, J. (1853): Kombinatorische Aufgabe. *J. Reine Angew. Math.,* **45**, 181-182.